MA 261 Fall 2013 Exam 2 Type 1

1. (**9 points**) Evaluate

$$\int_0^2 \int_0^{y^2} 3y^3 e^{xy}\, dx dy =$$

A. $e^4 - 4$

B. $e^6 - 5$

C. $e^8 - 8$

D. $e^6 - 6$

E. $e^8 - 9$

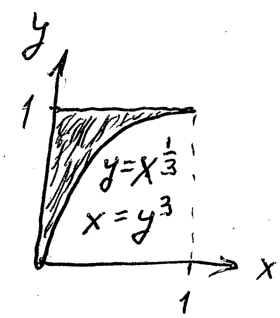
$$\int_0^2 3y^2 e^{xy}\Big|_{x=0}^{x=y^2} dy =$$

$$\int_0^2 3y^2 (e^{y^3} - 1)\, dy =$$

$$e^{y^3} - y^3 \Big|_0^2 = e^8 - 9$$

2. (**9 points**) Reverse the order of integration and evaluate

$$\int_0^1 \int_{x^{1/3}}^1 \frac{dydx}{y^4+1} =$$



A. $\dfrac{\ln 2}{4}$

B. $\ln 2$

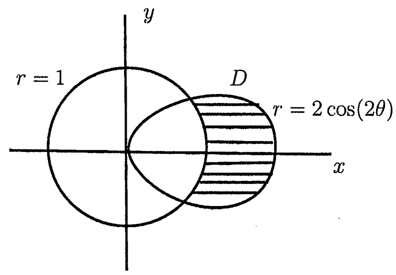
C. $4 \ln 2$

D. $4(\ln 2 - 1)$

E. $\ln 2 - 1$

$$\int_0^1 \int_0^{y^3} \frac{dx\, dy}{y^4+1} =$$

$$\int_0^1 \frac{y^3 dx}{y^4+1} = \frac{1}{4} \ln (y^4+1)\Big|_0^1 = \frac{\ln 2}{4}$$



3. (**9 points**) Which integral represents the area of D between the curves $r = 1$ and $r = 2\cos(2\theta)$ shown in the figure above?

A. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{1}^{2\cos 2\theta} r\,dr\,d\theta$

B. $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{1}^{2\cos 2\theta} r\,dr\,d\theta$

C. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{2\cos 2\theta} r\,dr\,d\theta$

D. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{1}^{2\cos 2\theta} r\,dr\,d\theta$

E. $\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \int_{1}^{2\cos 2\theta} r\,dr\,d\theta$

Intersection points of the two curves:

$$1 = 2\cos(2\theta)$$
$$2\theta = \pm \frac{\pi}{3}$$
$$\theta = \pm \frac{\pi}{6}$$

4. (**9 points**) Find the volume of region bounded by $2x + y + z = 2$ and the first octant.

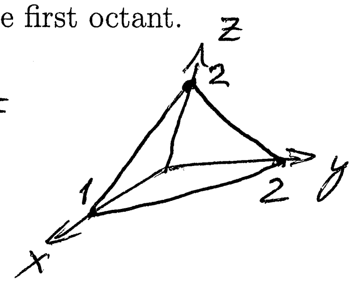
A. $\frac{2}{3}$

B. $\frac{4}{3}$

C. $\frac{3}{2}$

D. $\frac{8}{5}$

E. $\frac{3}{4}$



$$V = \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} dz\,dy\,dx =$$

$$= \int_0^1 \int_0^{2-2x} (2-2x-y)\,dy\,dx =$$

$$= \int_0^1 \left[(2-2x)^2 - \frac{(2-2x)^2}{2} \right] dx = -\frac{(2-2x)^3}{12}\Big|_0^1 = \frac{2}{3}$$

Alternatively, the volume of a pyramid with the base area 1 and height 2 is $\frac{1 \cdot 2}{3} = \frac{2}{3}$.

5. **(9 points)** Which integral gives the area of the surface defined by $z = \sin(x)\cos(y) + 4$ bounded by $x^2 + y^2 = 1$?

$$\frac{\partial z}{\partial x} = \cos x \cos y$$

$$\frac{\partial z}{\partial y} = -\sin x \sin y$$

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\; dA$$

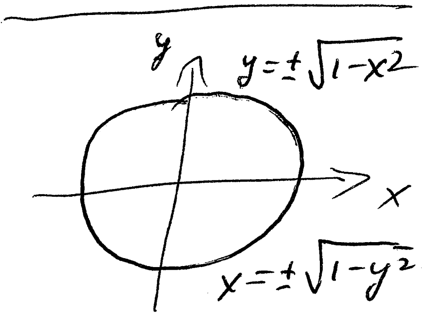
A. $\int_{-1}^{1}\int_{-1}^{1} \sqrt{\cos^2(x)\cos^2(y) + \sin^2(x)\sin^2(y) + 1}\; dy\, dx$

B. $\int_{-1}^{1}\int_{-1}^{1} \sqrt{\cos^2(y)\cos^2(y) + \sin^2(x)\sin^2(x) + 1}\; dy\, dx$

C. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{\cos^2(x)\cos^2(y) + \sin^2(x)\sin^2(y) + 1}\; dy\, dx$

D. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\cos^2(x)\cos^2(y) + \sin^2(x)\sin^2(y) + 1}\; dy\, dx$

E. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\cos^2(x)\cos^2(y) + \sin^2(x)\sin^2(y) + 1}\; dx\, dy$



In A and B, domain is not a circle

In C and E, dx and dy are in wrong order

6. **(9 points)** Compute $\iiint_E x\, dV$ where E is the region bounded by $z = \underbrace{5(x^2+y^2) - 1}_{5r^2-1}$, $z = \underbrace{4\sqrt{x^2+y^2}}_{4r}$, $\underbrace{x \geq 0, \text{ and } y \geq 0}_{0 \leq \theta \leq \frac{\pi}{2}}$.

A. $\frac{1}{5}$

B. $\frac{1}{2}$

C. $\frac{5}{3}$

D. $\frac{1}{3}$

E. $\frac{2}{3}$

At $x=y=0$, $5r^2-1 < 4r$, thus $5r^2-1$ is the lower bound and $4r$ is the upper bound. The intersection $5r^2-1 = 4r$ is at $r=1$. Thus

$$\iiint_E x\, dV = \int_0^{\pi/2}\int_0^1\int_{5r^2-1}^{4r} \underbrace{r\cos\theta}_{x}\, \underbrace{r\, dz\, dr\, d\theta}_{dV} = \int_0^1 (4r^3 - 5r^4 + r^2)\, dr = 1/3$$

Note that $\int_0^{\pi/2} \cos\theta\, d\theta = 1$

7. **(9 points)** A solid ball of radius 1 centered at the origin has the density $\underbrace{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}_{= G}$.
Find the mass of the ball.

A. $\frac{\pi^2}{2}$

B. $\frac{\pi^2}{3}$

C. $\frac{\pi^2}{4}$

D. $\frac{\pi^2}{5}$

E. $\frac{\pi^2}{6}$

$$\sqrt{x^2+y^2} = r = \rho \sin\phi$$

$$\sqrt{x^2+y^2+z^2} = \rho$$

$$m = \iiint_E \underbrace{G}_{\text{density}} \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \underbrace{\rho^2 \sin\phi}_{\text{density}} \; \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} =$$

$$2\pi \cdot \underbrace{\int_0^{\pi} \sin^2\phi \, d\phi}_{= \frac{\pi}{2}} \; \underbrace{\int_0^1 \rho^4 \, d\rho}_{= 1/5} = \frac{\pi^2}{5}$$

8. **(9 points)** Evaluate the line integral

$$\int_C (y^2+z^2)\, ds \text{ where}$$

$$C = \{x = t,\ y = \cos(2t),\ z = \sin(2t),\ 0 \le t \le 2\pi\}.$$

A. 2π

B. $2\pi\sqrt{2}$

C. $2\pi\sqrt{5}$

D. $\pi\sqrt{2}$

E. $\pi\sqrt{5}$

$$y^2 + z^2 = 1 \text{ on } C$$

$$ds = \sqrt{x'^2 + y'^2 + z'^2}\, dt = \sqrt{5}\, dt$$

$$\int_C (y^2+z^2)\, ds = \int_0^{2\pi} \sqrt{5}\, dt = 2\pi\sqrt{5}$$

9. (**9 points**) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = \langle x,\ y,\ xy \rangle$$

and C is parametrized by $\vec{r}(t) = \langle \cos t,\ \sin t,\ t \rangle$, with $0 \le t \le \pi/2$.

A. $\frac{3}{2}$

B. 1

C. $\frac{1}{2}$

D. 0

E. $-\frac{1}{2}$

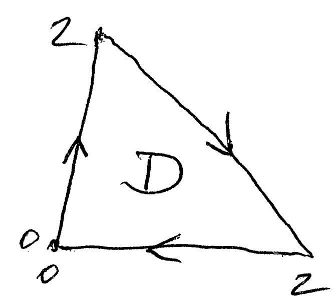
$dx = -\sin t\, dt, \quad dy = \cos t\, dt, \quad dz = dt$

$$\int_0^{\pi/2} \left(-\cos t \sin t + \sin t \cos t + \cos t \sin t\right) dt$$

$$= \frac{\sin^2 t}{2} \Big|_0^{\pi/2} = \frac{1}{2}$$

10. (**9 points**) Use Green's theorem to evaluate the line integral

$$\oint_C (x^2 + 4y)\, dx + (x - 3y^2)\, dy$$



where C is the triangle from $(0,0)$ to $(0,2)$ to $(2,0)$ and to $(0,0)$. Notice the orientation of the curve.

A. 12

B. -12

C. -6

D. 6

E. -4

$P = x^2 + 4y, \quad Q = x - 3y^2$

$Q_x - P_y = -3$

$$\oint_C P\,dx + Q\,dy = -\iint_D (Q_x - P_y)\, dx\,dy =$$

$$= 3\ \text{Area}(D) = 6$$

Note that $C = -\partial D$ (opposite orientation)

11. (**10 points**) Find a function $f(x, y, z)$ such that

$$\nabla f(x, y, z) = \vec{F} = (2y + 2xz^2)\vec{\imath} + (2x + 2yz^2)\vec{\jmath} + 2(x^2z + y^2z)\vec{k}$$

and use it to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{r}(t) = (t+1)\vec{\imath} + (2t-1)\vec{\jmath} + t^3\vec{k}$, $0 \le t \le 1$.

A. 4

B. 7

C. 8

D. 5

E. 11

$$f_x = 2y + 2xz^2 \Rightarrow f = 2xy + x^2z^2 + g(y,z)$$

$$f_y = 2x + g_y = 2x + 2yz^2 \Rightarrow g_y = 2yz^2$$

$$\Rightarrow g = y^2z^2 + h(z)$$

$$f_z = 2x^2z + g_z = 2x^2z + 2y^2z + h_z = 2x^2z + 2y^2z$$

$$\Rightarrow h_z = 0 \Rightarrow h \equiv \text{const}, \ \underline{\text{assume}} \ h = 0$$

Then $f = 2xy + x^2z^2 + y^2z^2$, and

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(2, 1, 1) - f(1, -1, 0)$$

$$= 9 - (-2) = 11$$