MATH 261 – FALL 2000 – SECOND EXAM– FORM 1
November 2, 2000

STUDENT NAME GRADING KEY

STUDENT ID

RECITATION HOUR

RECITATION INSTRUCTOR

INSTRUCTIONS:
1. This test booklet has 7 pages including this one.
2. Fill in your name, your student ID number, your recitation hour and your recitation instructor's name above.
3. There are 9 questions, each worth 11 points.
4. Questions 1 to 6 are multiple choice. Circle the letter of your choice for the correct answer. No partial credit will be given.
5. Questions 7 to 9 are partial credit. You should carefully explain your solution. No points will be given to solutions without explanations.
6. No books, notes or calculators may be used.

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1) The critical points of the function $f(x, y) = -x^2 + 4xy + y^2 - 2x$ are

A) $(1,1)$ and $(-1,2)$

B) $(\frac{1}{3}, -\frac{2}{3})$, $(\frac{1}{2}, -\frac{1}{4})$

C) $(-\frac{1}{5}, \frac{2}{5})$

D) $(-\frac{2}{5}, -\frac{3}{5})$

E) $(0,0)$ and $(-\frac{1}{5}, \frac{2}{5})$

$f_x = -2x + 4y - 2$

$f_y = 4x + 2y$

$f_x = 0$: $-2x + 4y - 2 = 0$

$f_y = 0$: $4x + 2y = 0$

$\downarrow$

$y = -2x$

$-2x + 4(-2x) - 2 = 0$

$-10x = 2$

$x = -\frac{1}{5}$ $\quad y = \frac{2}{5}$

cr. pl. $(-\frac{1}{5}, \frac{2}{5})$

2) The points $(0,0)$ and $(-\frac{2}{3}, \frac{4}{3})$ are critical for $f(x,y) = 4xy + 2x^2y - xy^2$. Which of the following is correct?

A) $(0,0)$ is a relative maximum and $(-\frac{2}{3}, \frac{4}{3})$ a relative minimum.

B) $(0,0)$ is a relative maximum and $(-\frac{2}{3}, \frac{4}{3})$ a saddle point.

C) $(0,0)$ is a saddle point and $(-\frac{2}{3}, \frac{4}{3})$ a relative minimum.

D) $(0,0)$ and $(-\frac{2}{3}, \frac{4}{3})$ are relative minima.

E) $(0,0)$ and $(-\frac{2}{3}, \frac{4}{3})$ are relative maxima.

$f_x = 4y + 4xy - y^2 \qquad f_y = 4x + 2x^2 - 2xy$

$f_{xx} = 4y \qquad f_{yy} = -2x \qquad f_{xy} = 4 + 4x - 2y$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (4y)(-2x) - (4+4x-2y)^2$

$D(0,0) = -16 < 0 \quad \therefore \boxed{(0,0) \text{ saddle}}$

$D(-\frac{2}{3}, \frac{4}{3}) = (4\cdot\frac{4}{3})(-2)(-\frac{2}{3}) - (4 - \frac{8}{3} - \frac{8}{3})^2 = \frac{64}{9} - \frac{16}{9} > 0$

$f_{xx}(-\frac{2}{3}, \frac{4}{3}) = 4\cdot\frac{4}{3} > 0 \quad \therefore \boxed{(-\frac{2}{3}, \frac{4}{3}) \text{ relative minimum}}$

3) The maximum of $f(x,y,z) = x + y + z$ subject to the constraint $(x-1)^2 + y^2 + z^2 = 1$ is

$g(x,y,z) = (x-1)^2 + y^2 + z^2$

A) $1 + \sqrt{3}$

B) $1 - \sqrt{3}$

C) $\sqrt{3}$

D) $1 + 2\sqrt{3}$

E) $1 + 3\sqrt{3}$

$(x-1)^2 + y^2 + z^2 = 1$

$\left.\begin{array}{l} 1 = \lambda\, 2(x-1) \\ 1 = \lambda\, 2y \\ 1 = \lambda\, 2z \end{array}\right] \rightarrow x - 1 = z, \quad y = z$

$\therefore z^2 + z^2 + z^2 = 1$

$z = \pm\frac{1}{\sqrt{3}}$

$z = \frac{1}{\sqrt{3}} \quad y = \frac{1}{\sqrt{3}}, \quad x = 1 + \frac{1}{\sqrt{3}} \qquad \boxed{(1+\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})}$

$z = -\frac{1}{\sqrt{3}} \quad y = -\frac{1}{\sqrt{3}} \quad x = 1 - \frac{1}{\sqrt{3}} \qquad \boxed{(1-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{3})}$

$f(1+\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = 1 + \frac{3}{\sqrt{3}} = 1 + \sqrt{3} \quad \leftarrow$ maximum

$f(1-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = 1 - \frac{3}{\sqrt{3}} = 1 - \sqrt{3}$

4) Evaluate $\int\int_R y\, dA$ where R is the region of the plane bounded by $x + y = 2$, $x = y$ and $y = 0$.

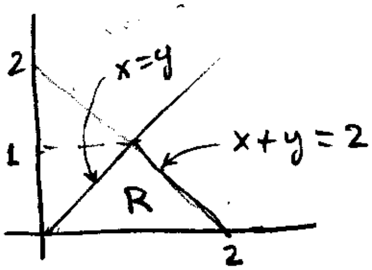
A) $\frac{3}{2}$

B) $\frac{1}{3}$

C) $\frac{1}{2}$

D) 1

E) 2



$R: \; y \le x \le 2-y, \quad 0 \le y \le 1$

$$\iint_R y\,dA = \int_0^1 \int_y^{2-y} y\,dx\,dy.$$

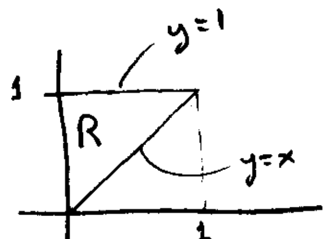
$$= \int_0^1 \left[yx\right]_{x=y}^{x=2-y} dy =$$

$$= \int_0^1 \left(y(2-y) - y^2\right) dy$$

$$= \int_0^1 (2y - 2y^2)\,dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

5) Interchange the order of integration to evaluate

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx.$$



$R: \; x \le y \le 1, \; 0 \le x \le 1$

$R: \; 0 \le x \le y, \; 0 \le y \le 1$

A) $2(e-1)$

B) $\frac{e}{2}$

C) $e+1.$

D) $e-1$

E) $\frac{e-1}{2}$

$$= \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$= \int_0^1 y e^{y^2}\,dy$$

$$= \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{e-1}{2}$$

6) Write an iterated double integral in polar coordinates, that expresses the area of the surface of the paraboloid $z = x^2 + y^2$, over the region in the xy plane given by $y \geq 0$ and $x^2 + y^2 \leq 1$.

A) $\int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \ \ r dr \ d\theta$

B) $\int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1 + 4r^2} \ \ r dr \ d\theta$

C) $\int_0^{\pi} \int_0^1 (1 + 2r) \ \ r dr \ d\theta$

D) $\int_0^{\pi} \int_0^1 \sqrt{1 + 4r^2} \ \ r dr \ d\theta$

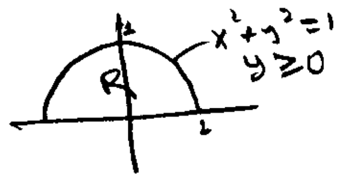
E) $\int_0^{\pi} \int_0^1 \sqrt{1 - 4r^2} \ \ r dr \ d\theta$

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

$$f(x,y) = x^2 + y^2$$

$$f_x = 2x \quad f_y = 2y.$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4(x^2+y^2)+1}$$

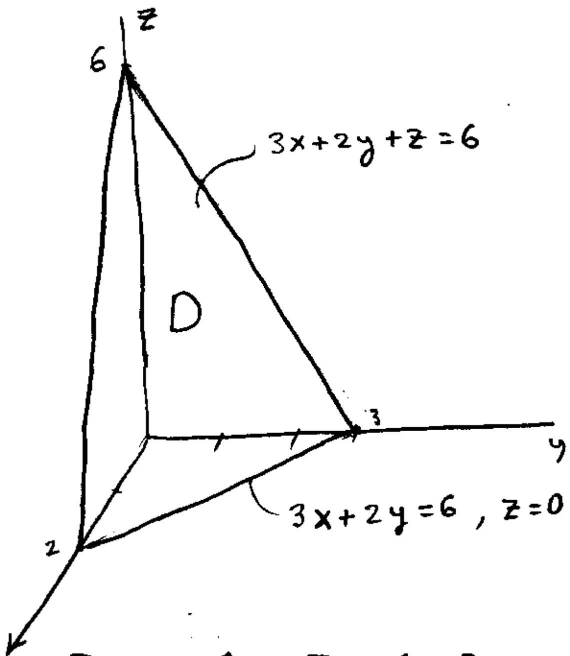


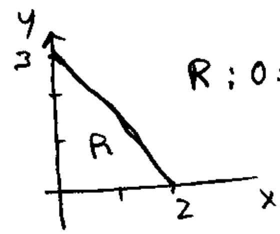
$R: \quad 0 \leq r \leq 1 \qquad 0 \leq \theta \leq \pi$

$$S = \int_0^{\pi} \int_0^1 \sqrt{4r^2+1} \ r dr d\theta$$

Remark: Questions 7, 8 and 9 require detailed solutions. No points will be given to answers without explanations. It is important to justify your steps. Even if you arrive at the correct answer, points will be deducted if your explanation is incorrect.

7) An object occupies the region in the first octant bounded above by the plane $3x + 2y + z = 6$ and by the planes $x = 0$, $y = 0$ and $z = 0$. If the mass density at the point (x, y, z) in the object is $\delta(x, y, z) = xyz$, set up a triple iterated integral that gives the total mass of the object. **It is not necessary to compute the integral. Please write your answer in the box.**





$R:\ 0 \le y \le 3 - \frac{3}{2}x$

$0 \le x \le 2$

$D:\quad 0 \le z \le 6 - 3x - 2y, \quad 0 \le y \le 3 - \frac{3}{2}x, \ 0 \le x \le 2$

$$m = \iiint_D \delta(x,y,z)\, dV$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} xyz\, dz\, dy\, dx$$

2 3 2 2 2

Answer:

$$m = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} xyz\, dz\,dy\,dx$$

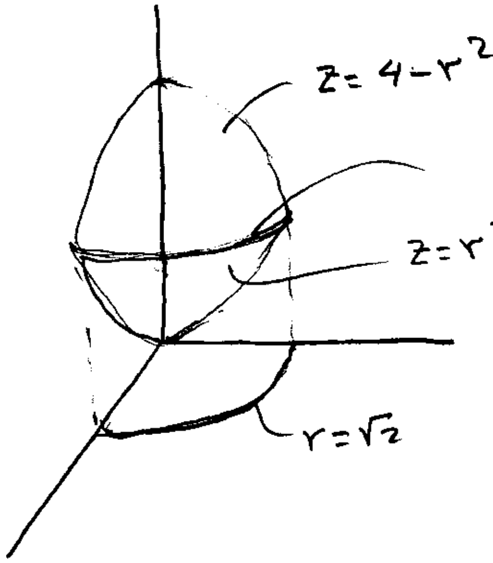
0 pts for problem if more than 2 items are wrong

$$\text{or } \int_0^3 \int_0^{2-\frac{2}{3}y} \int_0^{6-3x-2y} xyz\, dz\, dx\, dy$$

8) Let D be the part in the first octant of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Express

$$\int\int\int_D (x^2 + y^2)\, dV$$

as an iterated integral in cylindrical coordinates. **It is not necessary to compute the integral. Please write your answer in the box.**



$4 - r^2 = r^2 \rightarrow r^2 = 2 \quad r = \sqrt{2}$

$D: \; r^2 \le z \le 4 - r^2$
$0 \le r \le \sqrt{2}$
$0 \le \theta \le \frac{\pi}{2}$

$$\iiint_D (x^2+y^2)\, dV = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r^2 \, r \, dz\, dr\, d\theta$$

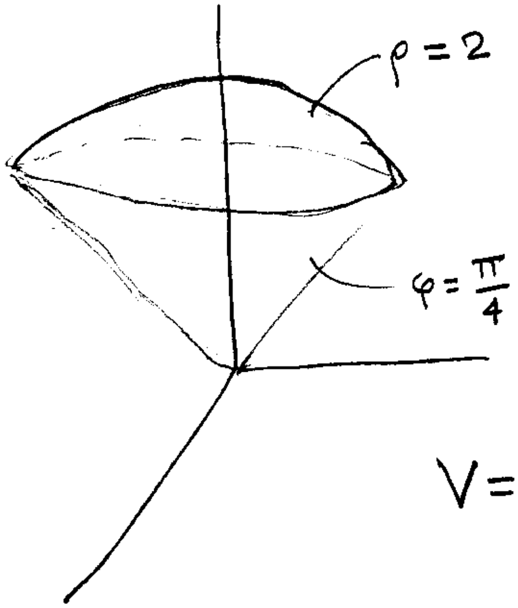
2 3 2 2 2

Answer:

$$\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r^3 \, dz\, dr\, d\theta$$

0 pts for problem if more than 2 items are wrong

9) Use spherical coordinates to set up a triple iterated integral that gives the volume of the region bounded from above by the surface $z = \sqrt{4 - x^2 - y^2}$ and below by the upper nappe of the cone $z^2 = x^2 + y^2$. **It is not necessary to compute the integral. Please write your answer in the box.**



$$D: \quad 0 \le \rho \le 2, \quad 0 \le \varphi \le \frac{\pi}{4}, \quad 0 \le \theta \le 2\pi$$

$$V = \iiint_D dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

2 2 2 3 2

Answer:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

0 pts for problems if more than 2 items are wrong