FALL 2017 MA 261 EXAM 1 SOLUTIONS

1. Suppose that the three points $(0,0,0)$, $(1,1,1)$, and (a,b,c) lie in a unique plane P. The line passing through $(0,0,0)$ and $(1,2,a)$ is normal to P. What is a?

A. $\boxed{a = -3}$

B. $a = 0$

C. $a = 1$

D. $a = 2$

E. $a = -1$

$\langle 1, 1, 1 \rangle \cdot \langle 1, 2, a \rangle = 0$

$1 + 2 + a = 0$

$a = -3$

2. Suppose a particle has acceleration

$$\mathbf{a}(t) = \frac{-1}{(1+t)^2}\mathbf{i} + \sqrt{t}\mathbf{j} + 2\mathbf{k}.$$

If the inital velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the initial position is $\mathbf{r}(0) = \mathbf{0}$, what is $\mathbf{r}(1)$?

A. $(\ln 2)\mathbf{i} + \frac{4}{15}\mathbf{j} + 2\mathbf{k}$

B. $\boxed{(\ln 2)\mathbf{i} + \frac{19}{15}\mathbf{j} + 2\mathbf{k}}$

C. $(1 + \ln 2)\mathbf{i} + \frac{4}{15}\mathbf{j}$

D. $(1 + \ln 2)\mathbf{i} + \frac{19}{15}\mathbf{j} + 2\mathbf{k}$

E. $\mathbf{i} + \frac{4}{15}\mathbf{j}$

$\vec{v} = \langle (1+t)^{-1}, \frac{2}{3}t^{3/2}, 2t \rangle + \langle 0, 1, 1 \rangle$

$\vec{r} = \langle \ln(1+t), \frac{4}{15}t^{5/2} + t, t^2 + t \rangle$

$\vec{r}(1) = $ B

3. Consider the space curve

$$\mathbf{r}(t) \;=(\cos t+\sin t)\mathbf{i}\;+\;(\cos t-\sin t)\mathbf{j}\;+\;t\mathbf{k}.$$

Compute $\mathbf{N}(t)$, the principal unit normal vector.

A. $(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}$

B. $\frac{1}{\sqrt{3}}(\cos t-\sin t)\mathbf{i}-\frac{1}{\sqrt{3}}(\cos t+\sin t)\mathbf{j}\;+\frac{1}{\sqrt{3}}\mathbf{k}$

C. $\boxed{-\frac{1}{\sqrt{2}}(\cos t+\sin t)\mathbf{i}+\frac{1}{\sqrt{2}}(\sin t-\cos t)\mathbf{j}}$

D. $\frac{1}{\sqrt{3}}(\cos t-\sin t)\mathbf{i}-\frac{1}{\sqrt{3}}(\cos t+\sin t)\mathbf{j}$

E. $\frac{1}{\sqrt{2}}(\cos t+\sin t)\mathbf{i}+\frac{1}{\sqrt{2}}(\sin t-\cos t)\mathbf{j}$

$\vec{r}\,' = \langle -\sin t+\cos t,\ -\sin t-\cos t,\ 1\rangle$

$|\vec{r}\,'| = \sqrt{\sin^2 t+\cos^2 t-2\sin t\cos t+\sin^2 t+\cos^2 t+2\sin t\cos t+1^2}$

$= \sqrt{3}$

$\vec{T}\,' = \frac{1}{\sqrt{3}}\langle -\cos t-\sin t,\ -\cos t+\sin t,\ 0\rangle$

$|\vec{T}\,'| = \frac{1}{\sqrt{3}}\sqrt{\cos^2 t+\sin^2 t+2\cos t\sin t+\cos^2 t+\sin^2 t-2\cos t\sin t+0^2} = \frac{1}{\sqrt{3}}\sqrt{2}$

$\vec{N} = \frac{\vec{T}\,'(\sqrt{3})}{\sqrt{2}} = C$

4. Consider the following two limits:

$$\text{I.}\quad \lim_{(x,y)\to(0,0)}\frac{xy}{x^2+xy+y^2} \qquad\qquad \text{II.}\quad \lim_{(x,y)\to(0,0)}\frac{x+y}{e^x+e^y}$$

A. $\boxed{\text{I. does not exist,}\quad \text{II. 0}}$

B. I. does not exist,   II. does not exist

C. I. 0,   II. does not exist

D. I. 0,   II. 0

E. I. 0,   II. 1

I. If $y=0$, $\lim_{x\to 0}\frac{0}{x^2+0}=0$

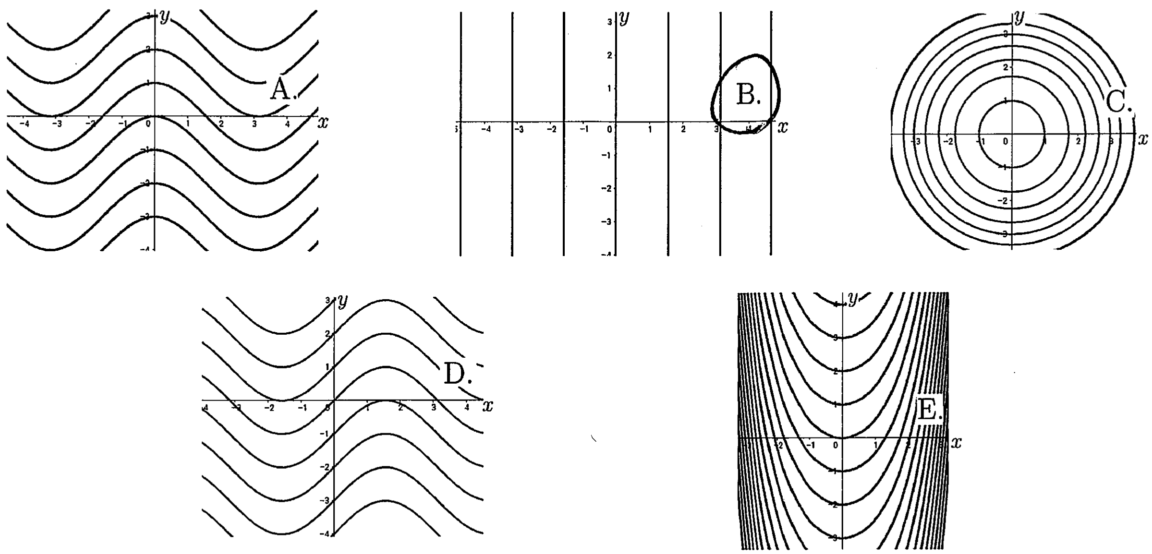
but if $y=x$, $\lim_{x\to 0}\frac{x^2}{x^2+x^2+x^2}=\frac{1}{3}$

DNE

II. $\frac{0+0}{e^0+e^0}=\frac{0}{2}=0.$

$\cos x = k \Leftrightarrow x = \cos^{-1} k$, VERTICAL LINES

5. Suppose $z = f(x, y) = \cos x$. Choose the correct contour map (level curves of f):



6. Suppose the function $f(x, y)$ has the following properties:

$$f_x(0,0) = 11, \quad f_x(1,1) = 3, \quad f_y(0,0) = 5, \quad f_y(1,1) = 7.$$

If $x = e^s - te^t$ and $y = e^t - se^s$, then what is f_s at $(s, t) = (0, 0)$?

A. -2

B. 10

C. 16

D. 6

E. $\boxed{-4}$

$x(0,0) = 1, \quad y(0,0) = 1.$

$x_s = e^s, \quad y_s = -e^s - se^s$

$x_s(0,0) = 1, \quad y_s(0,0) = -1$

$f_s = f_x(1,1)\, x_s(0,0) + f_y(1,1)\, y_s(0,0)$

$= (3)(1) + (7)(-1)$

f

x y

s

7. The length and width of a rectangle are measured as 10 cm and 4 cm, respectively, with an error in measurement of at most 0.1 cm each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

A. 5 cm^2

B. $\boxed{1.4 \text{ cm}^2}$

C. 0.5 cm^2

D. 4 cm^2

E. 0.4 cm^2

$A = xy$

$dA = y\,dx + x\,dy$

$= 4(0.1) + 10(0.1) = 1.4$

8. Find the point (x, y) at which f has a local minimum:

$$f(x,y) = \frac{1}{3}y^3 + x^2 + 4xy - 2x - 13y + 7.$$

A. $(1, -13)$

B. $(3, -1)$

C. $(1, -3)$

D. $\boxed{(-17, 9)}$

E. None of the above.

$f_x = 2x + 4y - 2 = 0 \rightarrow x = 1 - 2y$

$f_y = y^2 + 4x - 13$

$= y^2 + 4(1-2y) - 13$

$= y^2 - 8y - 9 = 0$

$= (y-9)(y+1)$

$f_{xx} = 2 > 0$

$f_{yy} = 2y$

$f_{xy} = 4$

$D = (2)(2y) - (4)^2 = 4y - 16$

If $y = -1$, $D < 0$ (saddle)

If $y = 9$, $D > 0$ and $x = 1 - 2(9) = -17$

9. The depth of a lake at the point (x, y) is $3x^2y + 5y^2$ feet. Assume that x and y are measured in miles. If a boat at the point $(2, 1)$ is sailing in the direction of the vector $3\mathbf{i} + \mathbf{j}$, at what rate is the water depth changing?

A. 12 feet/mile

B. $14\frac{1}{2}$ feet/mile

C. $\boxed{\dfrac{58}{\sqrt{10}} \text{ feet/mile}}$

D. 41 feet/mile

E. $\dfrac{23\sqrt{10}}{5}$ feet/mile

$\nabla f = \langle 6xy, 3x^2 + 10y \rangle$

$\nabla f(2,1) = \langle 12, 22 \rangle$

$\hat{u} = \frac{1}{\sqrt{3^2+1^2}} \langle 3, 1 \rangle$

$D_{\hat{u}} f = \frac{1}{\sqrt{10}} (12 \cdot 3 + 22 \cdot 1)$

10. Find $\dfrac{\partial z}{\partial x}$ at $(x, y, z) = (1, 2, 1)$, where z is defined implicitly as a function of x and y by the equation

$$\tan(\pi xyz) = x^2 - z^2.$$

A. $\dfrac{1}{\pi + 1}$

B. $1 - \pi$

C. $\dfrac{2 - 2\pi}{3}$

D. $\boxed{\dfrac{1 - \pi}{1 + \pi}}$

E. $\dfrac{\pi + 1}{1 - \pi}$

$\frac{\partial}{\partial x}(\tan \pi xyz) = \frac{\partial}{\partial x}(x^2 - z^2)$

$\sec^2(\pi xyz)\left[\pi yz + \pi xy \frac{\partial z}{\partial x}\right] = 2x - 2z\frac{\partial z}{\partial x}$

At $(1,2,1)$:

$\sec^2(2\pi)\left[2\pi + 2\pi \frac{\partial z}{\partial x}\right] = 2 - 2\frac{\partial z}{\partial x}$

$2\pi \frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial x} = 2 - 2\pi$

$\frac{\partial z}{\partial x} = \frac{1 - \pi}{\pi + 1}$

or, $F = \tan(\pi xyz) + z^2 - x^2$

implicit Function Theorem:

$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\left.\frac{\pi yz \sec^2(\pi xyz) - 2x}{\pi xy \sec^2(\pi xyz) + 2z}\right|_{(1,2,1)}$

$= -\frac{2\pi - 2}{2\pi + 2}$

11. Find the tangent plane to the level surface $xy^2z^3 = 12$ at $(3, 2, 1)$.

A. $x + 2y + 3z = 10$

B. $x + y + z = 6$

C. $3x + 2y + z = 14$

D. $x + 3y + 6z = 15$

E. $\boxed{x + 3y + 9z = 18}$

$F(x,y,z) = xy^2z^3$

$\nabla F = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla F(3,2,1) = \langle 2^2 \cdot 1^3, 2 \cdot 3 \cdot 2 \cdot 1^3, 3 \cdot 3 \cdot 2^2 \cdot 1^2 \rangle$

$= \langle 4, 12, 36 \rangle$

$\vec{n} = \langle 1, 3, 9 \rangle$

$\vec{v}(1) = \langle 1, 1, 1 \rangle$

12. Find $f'(1)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(1) = \langle 1, 1, 1 \rangle$, $\mathbf{u}'(1) = \langle 1, 2, 3 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

$\vec{v}\,'(1) = \langle 1, 2, 3 \rangle$

A. 6

B. 14

C. 28

D. $\boxed{12}$

E. 24

$f'(1) = \vec{u}(1) \cdot \vec{v}\,'(1) + \vec{u}\,'(1) \cdot \vec{v}(1)$

$= \langle 1,1,1 \rangle \cdot \langle 1,2,3 \rangle + \langle 1,2,3 \rangle \cdot \langle 1,1,1 \rangle$

$= 6 + 6$