NAME GRADING KEY

10–DIGIT PUID ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

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Page 2	/23
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Page 4	/29
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators, or any electronic devices may be used on this test.

4pts each NPC

(12) 1. Circle T if True or F if False. You do not need to show work.

(a) If $\lim_{n\to\infty} na_n = 5$, then $\sum_{n=1}^{\infty} a_n$ converges. T **F**

Compare with $\sum_{n=1}^{\infty}\frac{1}{n}$ which div. $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n a_n = 5 \therefore \sum a_n$ div

(b) There is a convergent alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$, with $b_n > 0$, and such that $\lim_{n\to\infty} b_n \neq 0$. $\lim_{n\to\infty}(-1)^n b_n \neq 0$. Test for divergence T **F**

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges. **T** F

If $\sum_{n=1}^{\infty}|a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ must be convergent.

$\boxed{12}$

(12) 2. Determine whether each of the following series is convergent or divergent. You do not need to show work.

4pts each. NPC

(a) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n}$ compare with $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent and use limit. comp. test

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} e^{1/n} = 1$

$\boxed{\text{divergent}}$

(b) $\sum_{n=1}^{\infty}\frac{2+\sin n}{n^2}$ compare with $\sum_{n=1}^{\infty}\frac{3}{n^2}$ which is conv.

$\frac{2+\sin n}{n^2} \le \frac{3}{n^2}$ for all n

$\boxed{\text{convergent}}$

(c) $\sum_{n=1}^{\infty}(2^{1/n}-1)^n$ Root test

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}(2^{1/n}-1) = 0 < 1$

$\boxed{\text{convergent}}$

$\boxed{12}$

(5) 3. $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4} \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} = \frac{3}{4} \frac{1}{1-\frac{3}{4}} = 3$

NPC

$\boxed{3}$

$\boxed{5}$

(27) 4. Determine whether each series is convergent or divergent. You must state the conditions of the test you are using and verify them if they are not obvious. Write your conclusion in the small box.

Each problem 4(a), 4(b), 4(c) is 9pts.
Look first for conv. or div. If wrong, 0 pts for problem
If right, check work and test
~~If there is no work → 0 pts for problem~~

(a) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{5^n n!}$

Show all necessary work here:

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{5^{n+1}(n+1)!}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}} \quad 3$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)5^n n!}{5^{n+1}(n+1)!\ 1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$= \underbrace{\frac{1}{5} \frac{2n+1}{n+1}}_{2} \xrightarrow[\text{as } n \to \infty]{} \frac{2}{5} < 1$$

1 1

By the 2 ratio test, the series is convergent $\boxed{9}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$

Show all necessary work here:

Alternating series test $b_n = \frac{1}{n5^n}$ 1

3 (i) $b_{n+1} \le b_n$ for all n ✓

3 (ii) $\lim_{n\to\infty} b_n = 0$ ✓

Or Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)5^{n+1}}}{\frac{1}{n5^n}} = \frac{n5^n}{(n+1)5^{n+1}}$$

$$= \frac{1}{5}\frac{n}{n+1} \xrightarrow[\text{as } n\to\infty]{} \frac{1}{5} < 1$$

∴ series is abs. conv. and hence conv.

By the 2 alternating series test, the series is convergent $\boxed{9}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{n^3+1}}$

Show all necessary work here:

Limit comparison test 3

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is divergent 1 (p-series $p=\frac{1}{2}<1$) 1

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2}{\sqrt{n^3+1}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^{3/2}+2n^{1/2}}{\sqrt{n^3+1}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\sqrt{1+\frac{1}{n^3}}} = 1 > 0$$ 1 1

By the limit comparison test, the series is divergent 2

9

(10) 5. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(10)^n}$ satisfies the conditions of the alternating series test.

(a) Write out the first five terms of the series.

$$-\frac{1}{10} + \frac{1}{2\cdot 10^2} - \frac{1}{3\cdot 10^3} + \frac{1}{4\cdot 10^4} - \frac{1}{5\cdot 10^5}$$ 5

−1 pt if only 1 term is wrong

(b) Use the alternating series estimation theorem to find the number of terms that we need to add in order to estimate the sum of the series with error < 0.0001.

$$\frac{1}{4\cdot 10^4} = \frac{1}{4}10^{-4} < 10^{-4} = 0.0001$$ 2

$$\frac{1}{3\cdot 10^3} = \frac{1}{3}10^{-3} = \frac{10}{3}10^{-4} > 10^{-4} = 0.0001$$ 3

3 terms

10

(5) 6. Find the Taylor series of the function $f(x) = e^x$ centered at $a = -1$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!}(x+1)^n$$

$f^{(n)}(x) = e^x$

$f^{(n)}(-1) = e^{-1}$ for all $n \geq 0$

1 if both limits are correct

4

$$e^x = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!}(x+1)^n$$

5

(14) 7. For the power series $\sum_{n=1}^{\infty} \frac{n}{5^n}(x+1)^n$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)(x+1)^{n+1}}{5^{n+1}}}{\frac{n(x+1)^n}{5^n}}\right| = \left|\frac{(n+1)(x+1)^{n+1}}{5^{n+1}}\frac{5^n}{n(x+1)^n}\right| =$

$= \overbrace{\frac{1}{5}\frac{n+1}{n}|x+1|}^{3} \xrightarrow[\text{as } n\to\infty]{} \frac{1}{5}|x+1|$ 1

$\therefore$ series conv if $\frac{1}{5}|x+1|<1$ or $|x+1|<5$ 2
and div if $|x+1|>5$

$\boxed{R = 5}$ 2

(b) The interval of convergence. (Don't forget to check the end points).

Series conv if $-6<x<4$

When $x=-6$: $\sum_{n=1}^{\infty}\frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty}(-1)^n n$ ← diverges 1 by the test for divergence 1 ($\lim_{n\to\infty}(-1)^n n$ DNE)

When $x=4$: $\sum_{n=1}^{\infty}\frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n$ ← diverges 1 by the test for divergence 1 ($\lim_{n\to\infty} n = \infty$)

Interval of convergence
$(-6, 4)$

1 1

14

(15) 8. For each function f find its Maclaurin series and radius of convergence. You may use known series to get your answer.

(a) $f(x) = \dfrac{1}{1-3x}$

1 if both limits are correct

$$\boxed{\frac{1}{1-3x} = \sum_{n=0}^{\infty} \overbrace{3^n x^n}^{3}, \quad R = \frac{1}{3}}$$ 1

5

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$, $R=1$

$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 3^n x^n$, $|3x|<1$ or $|x|<\frac{1}{3}$, $R=\frac{1}{3}$

(b) $f(x) = \dfrac{1}{(1+x)^2}$. (Hint: $\dfrac{d}{dx}\dfrac{1}{1+x} = -\dfrac{1}{(1+x)^2}$)

$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, $|x|<1$, $R=1$

1 3 1

$$\boxed{\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} \overbrace{(-1)^{n+1} n x^{n-1}}^{3}, \quad R = 1}$$

5

$-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$

$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \underset{m=n-1}{\overset{\text{or}}{=}} \sum_{m=0}^{\infty}(-1)^m (m+1) x^m \overset{\text{or}}{=} \sum_{n=0}^{\infty}(-1)^n (n+1) x^n$

(c) $f(x) = e^{-x^2}$

1 3 1

$$\boxed{e^{-x^2} = \sum_{n=0}^{\infty} \overbrace{(-1)^n \frac{x^{2n}}{n!}}^{3}, \quad R = \infty}$$

5

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, $R=\infty$

$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$