NAME GRADING KEY

10–DIGIT PUID

RECITATION INSTRUCTOR

RECITATION TIME

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TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators, or any electronic devices may be used on this test.

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(12) 1. Determine whether the following statements are true or false for any series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. (Circle T or F. You do not need to show work).

4 pts each
NPC

(a) If $0 < a_n < b_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges T F

(b) If $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges. T F

$\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} (-\frac{1}{n})$

(c) $\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} - (1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}) \right| \le \frac{1}{25}$ T F

Alternating series estimation theorem

$\boxed{12}$

(12) 2. Determine whether each of the following series is convergent or divergent. (You do not need to show work).

4 pts each
NPC

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}} \frac{2^n}{n!} = \frac{n+1}{2} \to \infty$

$\boxed{\text{divergent}}$

(b) $\sum_{n=1}^{\infty} \cos(\frac{1}{n^2})$

$\lim_{n\to\infty} \cos(\frac{1}{n^2}) = 1 \neq 0$
Test for divergence

$\boxed{\text{divergent}}$

(c) $\sum_{n=1}^{\infty} \frac{4n+1}{3n^3+2n+2}$

compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
and use Limit comparison test

$\boxed{\text{convergent}}$

$\boxed{12}$

(27) 3. Determine whether each series is convergent or divergent. You must verify that the conditions of the test you are using are satisfied and write your conclusion in the small box.

In problems 3(a)(b)(c) look first for conv. or div.
If wrong → 0 pts for problem
If right → check work and test
If there is no work → 0 pts for problem

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

Show all necessary work here:

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ (3) which is divergent (1) (p-series $p=\frac{1}{2}<1$) (1)

$\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$ for all n (2)

or $\lim_{n\to\infty} \frac{\frac{n+1}{n\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{3/2}+n^{1/2}}{n^{3/2}} = 1 > 0$ (1) (1)

By the limit comparison test, the series is divergent (2)

By the (2) comparison test, the series is divergent

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(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

Show all necessary work here:

Alternating Series test $b_n = \frac{n}{\sqrt{n^3+2}}$ (1)

(i) b_n decreasing?

Let $f(x) = \frac{x}{\sqrt{x^3+2}}$

$f'(x) \overset{(2)}{=} \frac{\sqrt{x^3+2} - x\,\frac{1}{2}\frac{1}{\sqrt{x^3+2}}\,3x^2}{x^3+2} = \frac{2(x^3+2)-3x^3}{(x^3+2)2\sqrt{x^3+2}}$

$= \frac{4-x^3}{2(x^3+2)^{3/2}} < 0$ for $x > \sqrt[3]{4}$

(1) $\therefore b_{n+1} < b_n$ for all $n \geq 2$ (1)

(ii) $\lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = \lim \frac{1}{\sqrt{n+\frac{2}{n^2}}} = 0$ (2)

By the (2) alternating series test, the series is convergent

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(c) $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$

Show all necessary work here:

Limit comparison test. 3

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent 1
(p-series $p=2>1$) 1

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$$

1 1

2

By the limit comparison test, the series is convergent

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(4) 4. $\sum_{n=0}^{\infty} \frac{3}{4^n} = 3 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 3 \frac{1}{1-\frac{1}{4}} = 4$

NPC

4

4

(6) 5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

Abs. conv. ? $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \frac{1}{4^{1/2}} + \cdots$

diverges 1 (p-series $p=\frac{1}{2}<1$) 1

Conv.? $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, $b_n = \frac{1}{\sqrt{n+1}}$

$b_{n+1} < b_n$ for all n

$\lim_{n\to\infty} b_n = 0$ 2 ~~conv~~ converges by alt. 1 series test

conditionally convergent 1

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(5) 6. Find the Taylor series of the function $f(x) = e^x$ centered at $a = 3$.

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

$f^{(n)}(x) = e^x$ $f^{(n)}(3) = e^3$ for all n

1 if both limits are correct

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$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$

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(14) 7. For the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}x^{n+1}}{n+2}}{\frac{(-1)^n x^n}{n+1}}\right| = \left|\frac{x^{n+1}}{n+2}\frac{n+1}{x^n}\right| = \underbrace{\frac{n+1}{n+2}}_{3}|x| \rightarrow |x|$ (1) as $n \to \infty$

$\therefore$ series converges if $|x|<1$ or $-1<x<1$, and diverges if $|x|>1$ (2)

$R = 1$ (2)

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(b) The interval of convergence. (Don't forget to check the end points).

When $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$ (2) div: p-series $p=1$

Whe $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ conv. by Alt. ser. test (2)

Interval of convergence (1) $(-1, 1]$ (1) or $-1<x\le 1$

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(20) 8. For each function f find its Maclaurin series and radius of convergence. You may use known series to get your answer.

(1) pt if both limits are correct

(a) $f(x) = \frac{1}{1+x}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$, $R=1$

$\therefore \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, $|-x|<1$ or $|x|<1$, $R=1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} \underbrace{(-1)^n x^n}_{3}$, $R = 1$ (1)

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(b) $f(x) = \frac{1}{(1-x)^2}$. (Hint: $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$)

$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$

$= \sum_{n=1}^{\infty} n x^{n-1}$ $|x|<1$

$R=1$

(1) $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} \underbrace{n x^{n-1}}_{3}$, $R = 1$ (1)

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(c) $f(x) = e^{3x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $R=\infty$

$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$, $R=\infty$

(1) $e^{3x} = \sum_{n=0}^{\infty} \underbrace{\frac{3^n}{n!} x^n}_{3}$, $R = \infty$ (1)

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(d) $f(x) = \sin x$.

(1) $\sin x = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{3}$, $R = \infty$ (1)

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