NAME GRADING KEY

10–DIGIT PUID

RECITATION INSTRUCTOR

RECITATION TIME

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TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators, or any electronic devices may be used on this test.

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(12) 1. Determine whether the following statements are true or false for any series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. (Circle T or F. You do not need to show work).

4 pts each
N PC

(a) If $0 < a_n < b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. $\sum_{n=1}^{\infty}\frac{1}{n}$ div. but $\sum_{n=1}^{\infty}\frac{1}{n^2}$ conv. T F

(b) If $0 < a_n$ for all n and $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n^3}} = 2$, then $\sum_{n=1}^{\infty} a_n$ converges. Compare with $\sum_{n=1}^{\infty}\frac{1}{n^3}$ limit comp. test T F

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges. By theorem: If $\sum a_n$ conv. abs. $\Rightarrow \sum a_n$ conv. T F

$\boxed{12}$

(12) 2. Determine whether each of the following series is convergent or divergent. (You do not need to show work).

4 pts each
NPC

(a) $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ is convergent geom. series, $r=\frac{1}{\sqrt{2}}$ and $\left|\frac{1}{\sqrt{2}}\right|<1$ $\boxed{\text{convergent}}$ $\boxed{4}$

(b) $\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{\pi}{3}\right)^n$ is divergent; geom. series $r=\frac{\pi}{3}>1$ $\boxed{\text{divergent}}$ $\boxed{4}$

(c) $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \sum_{n=0}^{\infty}\frac{2}{(n+1)(n+3)}$ convergent
compare with $\sum_{n=1}^{\infty}\frac{2}{n^2}$ which conv. p-series $p=2>1$ $\boxed{\text{convergent}}$ $\boxed{4}$

(30) 3. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

In problems 3(a)(b)(c) look first for conv. or div.
If wrong → 0 points for problem
If correct → check work and test
If there is no work → 0 pts for problem

Show all necessary work here:

Alternating series test $b_n = \frac{\ln n}{n}$ 1

(i) $\{b_n\}$ decreasing?

$f(x) = \frac{\ln x}{x}$, $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$ 2

$\therefore f'(x) < 0$ if $x > e$

$\therefore \{b_n\}$ is decreasing for $n \geq 3$ or 2

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ 2

$\frac{\infty}{\infty}$

3

By the alternating series test, the series is convergent

$\boxed{10}$

In problems 3(a),(b),(c) and 6(a)
If lim or → notation is wrong,
−1 pt for that problem

(b) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

Show all necessary work here:

Comparison test

Compare with $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ 2

which is convergent geom. series $r = \frac{1}{2}$, $\left|\frac{1}{2}\right| < 1$ 2

3

$\frac{n}{(n+1)\,2^{n-1}} < \frac{1}{2^{n-1}}$ for all $n \geq 1$

(or limit comparison test

last step: $\lim_{n\to\infty} \frac{\frac{n}{(n+1)2^{n-1}}}{\frac{1}{2^{n-1}}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$ 3)

3

By the comparison test, the series is convergent

$\boxed{10}$

or Ratio test:

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)}{(n+2)2^n} \cdot \frac{(n+1)2^{n-1}}{n} = \frac{1}{2}\frac{(n+1)^2}{n(n+2)} \to \frac{1}{2}$ as $n\to\infty$ 2 3

$\frac{1}{2} < 1$ 2 $\therefore$ ser. conv.

(c) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Show all necessary work here:

Limit comparison test 3

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent (p-series $p=1$) 2

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{y\to 0} \frac{\sin y}{y} = 1 > 0$$ 2

3

By the *limit comparison* test, the series is *divergent* $\boxed{10}$

(12) 4. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. (You do not need to show work).

4 pts each NPC

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n!}$

abs. conv.? : $\sum_{n=1}^{\infty} \frac{|\sin n|}{n!}$ conv?

absolutely convergent $\boxed{4}$

compare with $\sum_{n=1}^{\infty} \frac{1}{n!}$ which is conv. by ratio test:

$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1} \to 0 < 1$ as $n\to\infty$

(b) $\sum_{n=1}^{\infty} e^{-n} n!$

div. by ratio test

divergent $\boxed{4}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{-(n+1)}(n+1)!}{e^{-n}n!} = \frac{1}{e}(n+1) \to \infty$ as $n\to\infty$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

abs. conv.?: $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ conv?

conditionally convergent $\boxed{4}$

Div. by comparison with $\sum_{n=1}^{\infty} \frac{1}{n+1}$

conv.? yes by alt. ser. test

(16) 5. For the power series $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln n}}\right| = \frac{\ln n}{\ln(n+1)}|x| \xrightarrow{} |x|, \text{ as } n\to\infty$$ 4 2

$\therefore$ series conv. if $|x|<1$

or $-1<x<1$ 2

2

$\boxed{R = 1}$

(b) The interval of convergence. (Don't forget to check the end points).

At $x=-1$: $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ conv. by alt. ser. test 2

($b_n = \frac{1}{\ln n}$ decr and $b_n \to 0$ as $n\to\infty$)

At $x=1$: $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ div. by comp. with $\sum_{n=2}^{\infty} \frac{1}{n}$ 2

Interval of convergence

$[-1, 1)$

1 1

16

(9) 6. Evaluate the indefinite integral $\int \frac{t}{1-t^8}dt$ as a power series and give the radius of convergence.

$$\frac{1}{1-t^8} = \sum_{n=0}^{\infty} (t^8)^n, \quad |t^8|<1 \text{ or } |t|<1$$

$$\frac{t}{1-t^8} = \sum_{n=0}^{\infty} t^{8n+1}, \quad |t|<1$$

$$\int \frac{t}{1-t^8}dt = \int\left(\sum_{n=0}^{\infty} t^{8n+1}\right)dt$$

$$= \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C, \quad |t|<1$$

1 4 1 2

$\boxed{\int \frac{t}{1-t^8}dt = \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C, \quad R = 1}$

1

9

(9) 7. Find the first three nonzero terms of the Taylor series for $f(x) = \ln x$ centered at $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots$$

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$

3 3 3

$\boxed{\ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2}$

9