NAME GRADING KEY

10 DIGIT PUID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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3 pts each. NPC

(9) 1. Determine whether the following statements are true or false for any series $\sum_{n=1}^{\infty} a_n$. (Circle T or F. You do not need to show work).

(a) If $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent. By theorem, If $\sum a_n$ is conv. $\Rightarrow \lim_{n\to\infty} a_n = 0$ **T** F

(b) If $0 \le a_n \le \frac{1}{n\sqrt{n}}$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is conv. **T** F

(c) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. $\sum_{n=1}^{\infty} \frac{1}{n}$ is div. and $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$ T **F**

$\boxed{9}$

(12) 2. Determine whether each of the following series is convergent or divergent. (You do not need to show work).

4 pts each NPC

(a) $\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$ compare with $\sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is div.

$\frac{n+2}{n^2-1} \ge \frac{n}{n^2}$ for all n

$\boxed{\text{divergent}}$ $\boxed{4}$

(b) $\sum_{n=1}^{\infty} \frac{5(-2)^{n+1}}{3^n} = 5(-2)\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$ which is geom. series with $r = -\frac{2}{3}$, $\left|-\frac{2}{3}\right| < 1$

$\boxed{\text{convergent}}$ $\boxed{4}$

(c) $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3} = 5\sum_{n=1}^{\infty} \frac{1}{n^3} - 2\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

and each series on the right is conv. p-series with $p > 1$.

$\boxed{\text{convergent}}$ $\boxed{4}$

See added page 5 for alternate solution of 6(a)

In problems 3(a),(b),(c) and 6(a): If lim or $\to$ notation is wrong, -1 pt for that problem

(27) 3. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-\frac{n}{3}}$

For problems 3(a),(b),(c) look first for div. or conv.
If wrong $\to$ 0 points for problem
If correct $\to$ check work and test
If there is no work $\to$ 0 points for problem

Show all necessary work here:

Alternating series test $\quad b_n = n e^{-\frac{n}{3}}$ 1

(i) b_n decreasing?

$f(x) = x e^{-\frac{x}{3}}$; $f'(x) = x\left(-\frac{1}{3}\right)e^{-\frac{x}{3}} + e^{-\frac{x}{3}}$
$= e^{-\frac{x}{3}}\left(1 - \frac{x}{3}\right)$ 2

$f'(x) < 0$ if $x > 3$

$\therefore$ b_n decreasing for $n \geq 3$ or 2

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{n}{e^{n/3}} = \lim\limits_{x\to\infty} \frac{x}{e^{x/3}} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{1}{\frac{1}{3}e^{x/3}} = 0$ 2

$\frac{\infty}{\infty}$

By the alternating series test, the series is convergent 2

9

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+5}}{n^3+1}$

Show all necessary work here:

Compare with $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ 2
which is conv. (p-series, $p = 2 > 1$) 1

Limit comparison test

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\frac{\sqrt{n^2+5}}{n^3+1}}{\frac{1}{n^2}} =$ 2

$= \lim\limits_{n\to\infty} \frac{n^2\sqrt{n^2+5}}{n^3+1} = \lim\limits_{n\to\infty} \frac{\sqrt{1+\frac{5}{n^2}}}{1+\frac{1}{n^3}} = 1 > 0$ 2

By the limit comparison test, the series is convergent 2

9

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Show all necessary work here:

Integral test: $\int_2^{\infty} \frac{1}{x \ln x}\,dx$

$f(x) = \frac{1}{x \ln x}$ is continuous, positive, and decreasing in $[2,\infty)$

$\int_2^{\infty} \frac{1}{x \ln x}\,dx$ ② $= \lim_{t\to\infty}$ ② $\int_2^{t} \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \ln(\ln x)\Big|_2^t$

$= \lim_{t\to\infty} [\ln(\ln t)$ ② $- \ln(\ln 2)] = \infty$ ①

$\therefore$ integral is divergent and $\therefore$ series is div.

②

By the integral test, the series is divergent $\boxed{9}$

(10) 4. Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos(\frac{n}{2})}{n^2+4n}$ is absolutely convergent, conditionally convergent or divergent.

abs. conv.? : $\sum_{n=1}^{\infty} \left|\frac{\cos(\frac{n}{2})}{n^2+4n}\right|$ ③ conv.?

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ② which is conv. (p-series $p=2>1$)

$\left|\frac{\cos(\frac{n}{2})}{n^2+4n}\right| \leq \frac{1}{n^2}$ for all n ②

absolutely convergent ③ $\boxed{10}$

(9) 5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n}$.

(a) Write out the first five terms of the series.

$\frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} + \frac{1}{4^5}$ or ④ -1 pt if only 1 term is wrong

$\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \frac{1}{1024}$

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

$\frac{1}{64} > 0.01 > \frac{1}{256}$

⑤ NPC

3

(16) 6. For the power series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-2)^{n+1}x^{n+1}}{\sqrt[4]{n+1}}}{\frac{(-2)^n x^n}{\sqrt[4]{n}}}\right| = 2\sqrt[4]{\frac{n}{n+1}}\,|x| \longrightarrow 2|x|, \text{ as } n\to\infty$$ 4 2

$\therefore$ series conv if $2|x|<1$

or $|x|<\frac{1}{2}$

2

$R = \frac{1}{2}$

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if $-\frac{1}{2} < x < \frac{1}{2}$ 2

When $x=-\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ div. (p-series, $p=\frac{1}{4}<1$) 2

When $x=\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$

Conv. by Alt. ser. test $b_n=\frac{1}{n^{1/4}}$ 2

Interval of convergence

$\left(-\frac{1}{2}, \frac{1}{2}\right]$

or $-\frac{1}{2}<x\le\frac{1}{2}$

2 1

16

(10) 7. Find the power series representation of $\frac{x}{1-2x}$ (about $a=0$) and give its interval of convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n, \quad |2x|<1 \text{ or } |x|<\frac{1}{2}$$

$$\frac{x}{1-2x} = \sum_{n=0}^{\infty} 2^n x^{n+1}, \quad |x|<\frac{1}{2}$$

5

1

$\frac{x}{1-2x} = \sum_{n=0}^{\infty} 2^n x^{n+1}$

1

Interval of convergence

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ 3

or $-\frac{1}{2}<x<\frac{1}{2}$

10

(7) 8. Find the Taylor series for $f(x)=e^x$ centered at $a=3$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

$f(x)=e^x$

$f^{(n)}(x)=e^x$ for all n

$\therefore f^{(n)}(3)=e^3$ for all n

5

1

$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$

1

7

(27) 3. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-\frac{n}{3}}$

Show all necessary work here:

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \overset{1}{=} \lim_{x\to\infty} \overset{2}{\left| \frac{(-1)^n (n+1) e^{-\frac{n+1}{3}}}{(-1)^{n-1} n e^{-\frac{n}{3}}} \right|}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \frac{1}{e^{1/3}} = \frac{1}{e^{1/3}} \; 2 \; \overset{2}{<} 1$$

∴ series is absolutely convergent and hence it is convergent

By the 2 ratio test, the series is convergent

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+5}}{n^3+1}$

Show all necessary work here:

By the test, the series is