NAME Grading Key

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

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TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(9) 1. Determine whether the following statements are true or false (circle T or F). (You do not need to show work for this problem).

NPC

3×3

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. False : ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ T F $\boxed{3}$

(b) If $0 \le a_n \le b_n$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges. T F $\boxed{3}$

True by Comparison Test

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges. T F $\boxed{3}$

True : Contrapositive If $\sum |a_n|$ conv., then $\sum a_n$ co

(9) 2. Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent. (You do not need to show work for this problem).

NPC

3×3

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ $\sum (-1)^n \frac{1}{n}$ conv. by Alt. Ser. Test. while $\sum \frac{1}{n}$ divergent. $\boxed{\text{conditionally convergent}}$ $\boxed{3}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$ $0 \le \left|\frac{\sin(2n)}{n^2}\right| \le \frac{1}{n^2}$ $\sum \frac{1}{n^2}$ convergent $\boxed{\text{absolutely convergent}}$ $\boxed{3}$

(c) $\sum_{n=1}^{\infty}(-1)^n \sqrt{n}$ $\boxed{\text{divergent}}$ $\boxed{3}$

$\lim (-1)^n \sqrt{n}$ does not exist. Test for divergence.

(27) 2. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

9 x 3

(a) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+4}}$.

★ For problems 2 (a), (b), (c), look first for [ord co...]
If wrong → 0 pts for the problem.
If correct → check work and test

In problems 2(a),(b),(c) and 6(a):
If lim or → notation is wrong -1 pt for that problem.

★ No cre... if ther... no wor... (even ... the ti... answer corre...

Show all necessary work here :

Limit Comparison Test with

$a_n = \frac{n}{\sqrt{n^3+4}}$ and $b_n = \frac{n}{\sqrt{n^3}} = \frac{1}{n^{\frac{1}{2}}}$ 2

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+4}}}{\frac{n}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+4}} = 1$ 2

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ divergent 2

(p-series $p = \frac{1}{2} < 1$)

3 | By the Limit Comparison test, the series is divergent | 9

(b) $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$.

Show all necessary work here :

Ratio Test 2

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{5^{n+1}}}{\frac{n^3}{5^n}}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{5} = \frac{1}{5} < 1.$ 2 2

3 | By the Ratio test, the series is convergent | 9

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$.

Show all necessary work here :

Alternating Series Test. $b_n = \frac{\ln n}{n}$. ①

(i) b_n decreasing? Let $f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$ ②

$f'(x) < 0$ if $x > e$ $\therefore$ (i) ✓ for $n \geq 3$ ①

(ii) $\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$ ②

③ By the Alternating Series test, the series is convergent

$\boxed{9}$

(12) 4. Find the sum of the series if it is convergent or write "divergent" in the box. No partial credit.

NPC

4×3

(a) $\sum_{n=0}^{\infty} \frac{3(-4)^n}{5^{n+1}} = \frac{3}{5}\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n = \frac{3}{5}\left\{1 + \left(-\frac{4}{5}\right) + \left(-\frac{4}{5}\right)^2 + \cdots\right\}$.

geometric series with $a = \frac{3}{5}$ $r = -\frac{4}{5}$ $= \frac{3/5}{1-\left(-\frac{4}{5}\right)} = \frac{3}{9} = \frac{1}{3}$

$\boxed{\frac{1}{3}}$ $\boxed{4}$

(b) $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty}\left(-\frac{6}{5}\right)^{n-1} = 1 + \left(-\frac{6}{5}\right) + \left(-\frac{6}{5}\right)^2 + \left(-\frac{6}{5}\right)^3 + \cdots$

geometric series with $a = 1$ $r = -\frac{6}{5} < -1$.

$\boxed{\text{divergent}}$ $\boxed{4}$

(c) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$.

$= \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series with $p = \frac{1}{2} < 1$.

$\boxed{\text{divergent}}$ $\boxed{4}$

(8) 5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

4×2.

(a) Write out the first six terms of the series.

-1 pt for only one wrong term. or for all signs reversed.

$\boxed{-1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \text{ or } -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}}$ $\boxed{4}$

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

NPC

$\boxed{4}$ $\boxed{4}$.

$\frac{1}{24} > 0.01 > \frac{1}{120}$

(16) 6. For the power series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$, find the following, showing all work.

8×2 (a) The radius of convergence R.

Ratio Test (2)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)^2 5^{n+1}}}{\frac{|x|^n}{n^2 5^n}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}\frac{|x|}{5}$$

(4)

$$= \frac{|x|}{5}$$

$\therefore$ series converges if $\frac{|x|}{5} < 1$ i.e. $|x| < 5$

diverges if $\frac{|x|}{5} > 1$ i.e. $|x| > 5$

$R = 5$

(2) $\boxed{R = 5}$ 8

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if $-5 < x < 5$ (2)

When $x = -5$:

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ conv. } (p\text{-series } p = 2)$$ (2)

When $x = 5$:

$$\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ (abs.) conv.}$$ (2)

* consiste

Interval of convergence (2) $[-5, 5]$ 8

(10) 7(a). Find the power series representation for $\frac{1}{5-x}$.

3 + 7

$$\frac{1}{5-x} = \frac{1/5}{1-(\frac{x}{5})} = \frac{1}{5}\left\{1 + \left(\frac{x}{5}\right) + \left(\frac{x}{5}\right)^2 + \cdots\right\}$$

(1) (2)

$$\boxed{\frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{x}{5}\right)^n}$$ or $\frac{x^n}{5^{n+1}}$ 3

(b) Find the power series representation for $\frac{1}{(5-x)^2}$. (Hint: Use part (a) and differentiation).

(2)

$$\frac{1}{(5-x)^2} = \left(\frac{1}{5-x}\right)' = \left(\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\right)'$$

(2) (3)

$$\boxed{\frac{1}{(5-x)^2} = \sum_{n=1}^{\infty} \frac{n}{5^{n+1}} x^{n-1}}$$ ($n=1$ or $n=0$) 7

(9) 8. Find the first three nonzero terms of the Taylor series of $f(x) = \sqrt{x}$ centered at $a = 4$.

$$\begin{cases} f(x) = \sqrt{x} = x^{\frac{1}{2}} \\ f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \\ f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} \end{cases} \quad \begin{cases} f(4) = 2 \\ f'(4) = \frac{1}{4} \\ f''(4) = -\frac{1}{32} \end{cases}$$

(3) If ...

$$f(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \cdots$$

$$\boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$ 9

(2) (2) (2)