NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(14) 1. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following statements are true for any series $\sum_{n=1}^{\infty} a_n$? NPC

(I) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent. True by theorem

(II) If $\lim_{n\to\infty} |a_n| = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent. Not true. ex $\sum_{n=1}^{\infty} \frac{1}{n}$

(III) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. Not true. ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$

A. (I) only B. (I) and (II) only C. (II) only D. (III) only E. none $\boxed{7}$

(b) Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$, with any $p > 0$. Converges, by alternating series test

(II) $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^3-2n}$ Diverges. Comparison test, compare with $\sum_{n=1}^{\infty} \frac{1}{3n}$

(III) $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^3}$ Converges because it converges absolutely. Compare $\sum_{n=1}^{\infty} \left|\frac{\sin 2n}{n^3}\right|$ with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

A. (I) only B. (II) only C. (III) only D. (I) and (III) only E. all $\boxed{7}$

2. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

(8) (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

For problems 2(a), 2(b), and 2(c) look first for conv. or div.
If wrong → 0 pts for problem
If right → check work and test

Show all necessary work here:

Integral test. Let $f(x) = \frac{1}{x \ln x}$ $x \in [2, \infty)$

f is continuous, positive and decreasing for $x \in [2,\infty)$

and $f(n) = \frac{1}{n \ln n} = a_n$

$$\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \Big[\ln(\ln x)\Big]_2^t$$

4

$$= \lim_{t\to\infty} \left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$$

2

$\therefore \int_2^\infty \frac{1}{x \ln x}\,dx$ is divergent

2

By the integral test, the series is divergent

$\boxed{8}$

(8) (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$

Show all necessary work here:

Alternating series test $b_n = \frac{\sqrt{n}}{n+4}$

(i) b_n decreasing? Let $f(x) = \frac{\sqrt{x}}{x+4}$

$$f'(x) = \frac{(x+4)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+4)^2} = \frac{x+4-2x}{2\sqrt{x}\,(x+4)^2} = \frac{4-x}{2\sqrt{x}\,(x+4)^2}$$

2

$f'(x) < 0$ for $x > 4$ ← or 2

$\therefore$ b_n are decreasing for $n > 4$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$ 2

By the alternating series test, the series is convergent

2

$\boxed{8}$

(8) (c) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

Justify your answer and show all necessary work here:

$a_n = n \sin(\frac{1}{n})$

$\lim_{n\to\infty} a_n$ 4 $= \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ 2 $\neq 0$ 2

∴ series is divergent by theorem (or "test for divergence")

The series is divergent $\boxed{8}$

(8) 3. Find the sum of the series if it is convergent or write "divergent" in the box. No partial credit.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{4}\left(1 + (-\frac{1}{4}) + (-\frac{1}{4})^2 + \cdots\right)$

geometric series $= \frac{1}{4}\,\frac{1}{1-(-\frac{1}{4})} = \frac{1}{5}$

$\frac{1}{5}$ $\boxed{4}$

(b) $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{3^n} = \frac{1}{3}\sum_{n=1}^{\infty}\left(-\frac{4}{3}\right)^{n-1}$

geometric series divergent because $|-\frac{4}{3}| > 1$

divergent $\boxed{4}$

(8) 4. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$.

(a) Write out the first five terms of the series.

$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!}$ or 4

$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120}$

−1 pt for only one wrong term or for all signs reversed

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error $< \frac{1}{20}$

4 NPC

3 terms $\boxed{8}$

(10) 5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

−2pts if abs. value sign is not shown or not implied

Absolutely convergent? Use ratio test

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}}\right| = \frac{3^{n+1}\, n!}{(n+1)!\, 3^n} = \frac{3}{n+1}$ 4 $\xrightarrow[n\to\infty]{} 0 < 1$ 1 1 ∴ ser. is abs. conv.

4

0 pts for problem if answer in box is "conditionally convergent" or "divergent"

5 pts for problem if work is all correct and answer in box is "convergent"

The series is absolutely convergent $\boxed{10}$

(16) 6. For the power series $\sum_{n=1}^{\infty}(-1)^n n 4^n x^n$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)4^{n+1}x^{n+1}}{(-1)^n n 4^n x^n}\right| = \frac{n+1}{n} 4|x| \xrightarrow[n\to\infty]{} 4|x| \quad ④ \quad ②$$

0 points beyond this point if this is wrong

$\therefore$ series converges if $4|x|<1$

or $|x|<\frac{1}{4}$

② $\boxed{R = \frac{1}{4}}$ 8

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if $-\frac{1}{4}<x<\frac{1}{4}$ ②

When $x=-\frac{1}{4}$: $\sum_{n=1}^{\infty}(-1)^n n 4^n\left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} n$ diverges ②

When $x=\frac{1}{4}$: $\sum_{n=1}^{\infty}(-1)^n n 4^n\left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty}(-1)^n n$ diverges ②

Intervals of convergence ① ①
$\left(-\frac{1}{4}, \frac{1}{4}\right)$
or $-\frac{1}{4}<x<\frac{1}{4}$

8

(10) 7. (a) Find the power series representation of $-\frac{1}{5-x}$ (about $a=0$).

$$-\frac{1}{5-x} = -\frac{1}{5}\,\frac{1}{1-\frac{x}{5}} = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$$

$$\boxed{-\frac{1}{5-x} = -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}}$$ 3

② ① for limits

If answer in (a) has only a minor numerical or sign error, grade (b) with consistency

(b) Use integration of power series and the fact that $\ln(5-x) = -\int \frac{1}{5-x}\,dx$ for a certain value of the constant of integration C, to find the power series for $\ln(5-x)$.

$$\ln(5-x) = -\int\frac{1}{5-x}\,dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)} + C$$

Set $x=0$: $\ln 5 = C$

$$\boxed{\ln(5-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)} + \ln 5}$$ 7

① ① ③ ②

(10) 8. Find the Taylor series for $f(x)=\frac{1}{x}$ centered at $a=2$.

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = f(2) + \frac{f^{(1)}(2)}{1!}(x-2) + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \cdots$$

or ③

$f(x) = \frac{1}{x} = x^{-1}$ $\quad f(2) = 2^{-1}$

$f^{(1)}(x) = -1\,x^{-2}$ $\quad f^{(1)}(2) = -2^{-2}$

$f^{(2)}(x) = 1\cdot 2\, x^{-3}$ $\quad f^{(2)}(2) = 2\, 2^{-3}$

$f^{(3)}(x) = -1\cdot 2\cdot 3\, x^{-4}$ $\quad f^{(3)}(2) = -3!\, 2^{-4}$

$f^{(n)}(2) = (-1)^n n!\, 2^{-(n+1)}$

$$\therefore \frac{1}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n n!\, 2^{-(n+1)}}{n!}(x-2)^n$$

or $\frac{1}{x} = \frac{1}{2+(x-2)} = \frac{1}{2}\,\frac{1}{1+\frac{x-2}{2}}$ ③

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

① ⑤

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n}$$ 10

①