NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(15) 1. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following statements are always true for any series $\sum_{n=1}^{\infty} a_n$ with positive terms?

(I) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. Not true, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(II) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n$ converges. True. Root test

(III) If $\lim_{n\to\infty} \frac{a_n}{\frac{1}{\sqrt{n}}} = \frac{5}{6}$, then $\sum_{n=1}^{\infty} a_n$ diverges. True. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges and use limit comparison test

(IV) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. Not true. Ratio test inconclusive. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

A. (II) and (IV) only B. (I), (II) and (III) only C. (I) and (III) only
D. (II) and (III) only E. all

$\boxed{8}$

(b) Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$ (II) $\sum_{n=2}^{\infty} \frac{n}{(n-1)2^n}$

(III) $1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} - \dots$

(I) converges. Comparison test, compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(II) converges. Limit comparison test, compare with $\sum_{n=2}^{\infty} (\frac{1}{2})^n$

(III) converges. Alt. series test

A. (I) and (II) only B. (I) and (III) only C. (II) and (III) only D. (III) only
E. all

$\boxed{7}$

(20) 2. Determine whether each series is convergent or divergent. You must show all necessary work and write your conclusion in the small box.

For problems 2(a) and 2(b) look first for conv. or div.
If wrong → 0 pts for problem
If right → check work and test

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$

Show all necessary work here:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ 3 which diverges 1 (p-series, $p=1$) 1

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n(n+1)(n+2)}}}{\frac{1}{n}} \quad 1 \quad = \lim_{n\to\infty} \frac{n}{\sqrt[3]{n(n+1)(n+2)}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt[3]{1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}} = 1 > 0$$

1 1 (or $\neq 0$)

2

By the limit comparison test, the series is divergent

10

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n+1)!}$

Show all necessary work here: Ratio test. $a_n = \dfrac{3^n}{(2n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(2n+3)!}}{\frac{3^n}{(2n+1)!}} \quad 3$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{(2n+3)!}\,\frac{(2n+1)!}{3^n} = \lim_{n\to\infty} 3\,\frac{1}{(2n+2)(2n+3)} \quad 2$$

$$= 0 < 1$$

1 1

3

By the ratio test test, the series is Convergent

(10) 3. Consider the convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$.

(a) Write out the first six terms of the series.

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

or 4

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \cdots$$

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

$\frac{1}{120} < 0.01$ 3

$\frac{1}{24} > 0.01$

3 $\boxed{4}$

10

(10) 4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$ is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

absolutely convergent ?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$ convergent? Yes 3

(p-series, $p = \frac{3}{2} > 1$) 2

5

0 points for problem if answer is wrong

$\boxed{\text{absolutely convergent}}$

(9) 5. Find the sum of each series if it is convergent, or write divergent in the box. No partial credit.

(a) $\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n = \frac{1}{e^2} \cdot \frac{1}{1-\frac{1}{e^2}} = \frac{1}{e^2-1}$ $\boxed{\frac{1}{e^2-1}}$ 3

(b) $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$ $\boxed{\frac{2}{3}}$ 3

(c) $\sum_{n=1}^{\infty} \pi^{n-1}$ $\pi > 1$ $\boxed{\text{divergent}}$ 3

(*) -3 pts if abs. value sign is missing here and appears later on

(16) 6. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} n^3(x-5)^n$. Don't forget to test for convergence at the end points of the interval. You must show all work.

Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right|$ 6 (*)

$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 |x-5| = |x-5|$ 2 0 pts beyond this point if this is wrong

$\therefore$ Series converges for $|x-5|<1$ or $4<x<6$ 2

When $x=4$: $\sum_{n=1}^{\infty}(-1)^n n^3$ diverges 2 by the test for divergence ($\lim_{n\to\infty}(-1)^n n^3$ DNE)

when $x=6$: $\sum_{n=1}^{\infty} n^3$ diverges 2 ($\lim_{n\to\infty} n^3 = \infty \neq 0$).

$\therefore$ interval of convergence: $(4,6)$

$\boxed{1\ (4,\ 6)\ 1 \quad \text{or } 4<x<6}$ $\boxed{16}$

(10) 7. Evaluate the indefinite integral $\int \frac{1}{1+x^4}\,dx$ as a power series and determine its radius of convergence R.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$. $\frac{1}{1+x^4} = \sum_{n=0}^{\infty}(-x^4)^n = \sum_{n=0}^{\infty}(-1)^n x^{4n}$, $|-x^4|<1$ or $|x|<1$

$\int \frac{1}{1+x^4}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{4n}\right)dx$

$= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$, $|x|<1$

$\boxed{\int \frac{1}{1+x^4}\,dx = C + \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{x^{4n+1}}{4n+1}}_{6},\ R = 1}$ 1 1 2 $\boxed{10}$

(10) 8. Find the Taylor series for $f(x) = \frac{1}{x+1}$ centered at $a=1$.

$f(x) = \sum \frac{f^{(n)}(1)}{n!}(x-1) = f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots$ or 3 or

$f(x) = (x+1)^{-1}$ $\quad f(1) = 2^{-1}$

$f^{(1)}(x) = -(x+1)^{-2}$ $\quad f^{(1)}(1) = -2^{-2}$

$f^{(2)}(x) = 2(x+1)^{-3}$ $\quad f^{(2)}(1) = 2\cdot 2^{-3}$

$f^{(3)}(x) = -3\cdot 2(x+1)^{-4}$ $\quad f^{(3)}(1) = -3\cdot 2\cdot 2^{-4}$

$f^{(n)}(1) = (-1)^n n!\, 2^{-(n+1)}$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \frac{n!\,2^{-(n+1)}}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{n+1}}(x-1)^n$

or: $\frac{1}{x+1} = \frac{1}{2+(x-1)} = \frac{1}{2}\cdot\frac{1}{1+\frac{x-1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n$

$\boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{1}{2^{n+1}}(x-1)^n}_{5}}$ 1 1 $\boxed{10}$