NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

Page 1	/10
Page 2	/22
Page 3	/28
Page 4	/16
Page 5	/24
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, 4 and 5.
2. The test has five (5) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(10) 1. Evaluate the limit as a number, ∞, or $-\infty$. (You need not show work for this problem).

(a) $\lim\limits_{n\to\infty} \dfrac{2n^2-4}{-n-5} = \lim\limits_{n\to\infty}\left(-\dfrac{n^2}{n}\,\dfrac{2-\frac{4}{n^2}}{1+\frac{5}{n}}\right) = -\infty$ NPC

$\boxed{-\infty}$ [3]

(b) $\lim\limits_{n\to\infty} \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\sin\left(\dfrac{\pi n}{2n+3}\right)\right)$

$\lim\limits_{n\to\infty} \sin\left(\dfrac{\pi n}{2n+3}\right) = \lim\limits_{n\to\infty}\sin\left(\dfrac{\pi}{2+\frac{3}{n}}\right) = \sin\dfrac{\pi}{2} = 1.$

$\therefore \lim\limits_{n\to\infty} \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\sin\dfrac{\pi n}{2n+3}\right) = \cos^{-1}\dfrac{1}{\sqrt{2}} = \dfrac{\pi}{4}$

$\boxed{\dfrac{\pi}{4}}$ [4]

(c) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = \lim\limits_{x\to\infty} e^{x\ln(1+\frac{1}{x})} = e^{\lim\limits_{x\to\infty} x\ln(1+\frac{1}{x})}$

$\lim\limits_{x\to\infty} x\ln(1+\frac{1}{x}) = \lim\limits_{x\to\infty}\dfrac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \dfrac{0}{0}$

$\overset{L'H}{=} \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 1$

$\boxed{e}$ [3]

(12) 2. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following series converge?

NPC

(I) $\sum_{n=2}^{\infty} \frac{1-n^2}{1+n}$ div. $\left(\lim_{n\to\infty} \frac{1-n^2}{1+n} \neq 0\right)$

(II) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ conv. (alt. ser. test)

(III) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \dots = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ conv. (p-ser. $p = \frac{3}{2} > 1$)

A. (III) only B. none C. (II) only D. (II) and (III) only E. (I) and (II) only $\boxed{6}$

(b) Which of the following statements are true?

(I) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. no

(II) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. no

(III) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{3}{2}$, then $\sum_{n=1}^{\infty} a_n$ converges no

A. (I) only B. (II) only C. (III) only D. none E. (I) and III) only $\boxed{6}$

(10) 3. Let s be the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1+n+6n^2}$. What is the smallest number of terms you have to add up in order to approximate s with an error less than 0.01? You must show work.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1+n+6n^2}$$

By alternating series test, the series converges and the truncation error $E_j < a_{j+1}$

$$= \frac{1}{1+1+6} - \frac{1}{1+2+24} + \frac{1}{1+3+54} - \frac{1}{1+4+6\cdot 16} + \cdots$$

$$\frac{1}{1+4+6\cdot 16} = \frac{1}{101} < 0.01$$

If answer is wrong, give 5 points for attempting to compare the abs. value of a term with 0.01

$\boxed{3}$ $\boxed{10}$

-1pt each time "lim" and "=" are inconsistent.

(12) 4. (a) Prove that $\lim_{n\to\infty} n\tan\frac{1}{n} = 1$. (You may use $\lim_{x\to 0}\frac{\sin x}{x} = 1$).

$$\lim_{n\to\infty} n\tan\frac{1}{n} = \lim_{x\to\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}\frac{1}{\cos\frac{1}{x}}$$

$$= \lim_{y\to 0}\frac{\sin y}{y}\frac{1}{\cos y} = 1\cdot 1 = 1$$

Or L'H rule $= \lim_{x\to\infty}\frac{\sec^2(\frac{1}{x})(-\frac{1}{x^2})}{-\frac{1}{x^2}} = 1$

$\boxed{3}$

(b) Use (a) to determine whether the series $\sum_{n=1}^{\infty}\frac{\tan\frac{1}{n}}{n}$ is convergent or divergent.

You must show all work and name the test you are using, if any.

$$\sum_{n=1}^{\infty}\frac{\tan\frac{1}{n}}{n} = \sum_{n=1}^{\infty}\frac{n\tan\frac{1}{n}}{n^2}$$ 2

compare with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 2 which converges (p-series 1 $p=2>1$) 1

$$\lim_{n\to\infty}\frac{\frac{\tan\frac{1}{n}}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n\tan\frac{1}{n} = 1 \neq 0$$ 1

By the limit comparison test 1

The series is Convergent 1

$\boxed{9}$

(8) 5. Find the sum of the series $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n+1}$, if it converges.

$$\sum_{n=1}^{\infty} 5\left(\tfrac{1}{2}\right)^{n+1} = 5\left(\tfrac{1}{2}\right)^2 + 5\left(\tfrac{1}{2}\right)^3 + 5\left(\tfrac{1}{2}\right)^4 + \cdots$$

$$= 5\left(\tfrac{1}{2}\right)^2\left[1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \cdots\right]$$

$$= 5\cdot\frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} = \frac{5}{2}$$

If answer is wrong, give 3 points for attempting to sum using geometric series

$\frac{5}{2}$

$\boxed{8}$

(8) 6. Determine whether the series $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ diverges, converges conditionally, or converges absolutely. You must justify your answer.

Converges absolutely?

$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ 2 conv? Compare with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 1 which converges 1 (p-series, $p=2>1$)

$\left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$ 2 for $n = 1, 2, \ldots$

By comparison test 1

series conv. abs

1

The series converges absolutely

$\boxed{8}$

7. Determine whether each series is convergent or divergent. You must show all necessary work and write your conclusion in the small box.

(8) (a) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

For problems 7(a), 7(b) and 7(c): Look first at "conv" or "div". If wrong → 0pts for problem. If right, check work + test. If only test and conv or div are correct → 2 pts

Show all necessary work here:

$$a_n = \frac{n!}{e^n} \qquad a_{n+1} = \frac{(n+1)!}{e^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}} = \lim_{n\to\infty} \frac{(n+1)!\; e^n}{e^{n+1}\; n!}$$

3

$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1$$

2 1

2

By the ratio test, the series is divergent

8

(8) (b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Show all necessary work here:

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges (p-series, $p = 2 > 1$)

2 1 1

2

$$\frac{1}{n\sqrt{n^2+1}} \le \frac{1}{n^2} \quad \text{for all } n \ge 1$$

$$\left[\text{Or} \quad \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} = 1 \neq 0\right]$$

2

By the comparison test, the series is convergent

[or limit comparison]

8

(8) (c) $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

Show all necessary work here:

$$a_n = \left(\frac{n}{3n+1}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \overbrace{\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{3n+1}\right)^n}}^{③} = \lim_{n\to\infty} \frac{n}{3n+1} = \overset{②}{\frac{1}{3}} \overset{①}{<} 1$$

By the ② root test, the series is convergent $\boxed{8}$

(16) 8. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 2^n} x^n$. Don't forget to test for convergence at the end points of the interval. You must show all work.

Generalized ratio test. ④

$$\lim_{n\to\infty} \left| \frac{\frac{1}{\sqrt{n+1}\, 2^{n+1}} x^{n+1}}{\frac{1}{\sqrt{n}\, 2^n} x^n} \right| = \lim_{n\to\infty} \frac{\sqrt{n}\, 2^n}{\sqrt{n+1}\, 2^{n+1}} |x| = \overset{②}{\frac{1}{2}|x|}$$

$\therefore$ series converges (abs.) for $\frac{1}{2}|x| < 1$

or $|x| < 2$ or $-2 < x < 2$ ②

At $\underline{x=2}$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 2^n} 2^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges ② (p-series ① $p = \frac{1}{2} < 1$)

At $\underline{x=-2}$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 2^n} (-2)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges ②

① (alt. series test $\left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$)

$$\boxed{-2 \overset{①}{\leq} x \overset{①}{<} 2}$$