NAME GRADING KEY

10–DIGIT PUID

RECITATION INSTRUCTOR

RECITATION TIME

Page 1	/19
Page 2	/30
Page 3	/26
Page 4	/25
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators, or any electronic devices may be used on this test.

---

(9) 1. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x}\sin x\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos^2 x}{\cos^2 x}\sin x\,dx$ 4

$u=\cos x \quad du=-\sin x\,dx$

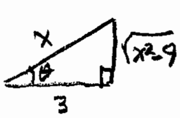
$x=0 \to u=1 \quad x=\frac{\pi}{4} \to u=\frac{1}{\sqrt{2}}$

$\displaystyle = \int_1^{\frac{1}{\sqrt{2}}} \frac{u^2-1}{u^2}\,du = \int_1^{\frac{1}{\sqrt{2}}} (1-u^{-2})\,du$

$\displaystyle = \left[u+\frac{1}{u}\right]_1^{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}+\sqrt{2}-2$ 5

$\boxed{\frac{1}{\sqrt{2}}+\sqrt{2}-2}$ 9

(10) 2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta\, 3\tan\theta}3\sec\theta\tan\theta\,d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$ 5



$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9} = 3\tan\theta$

$\displaystyle = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$ 3

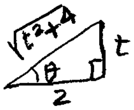
$\displaystyle = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$ 2

−2 pts if wrong number

−1 pt for missing +C (one time for test)
−1 pt for missing dx, du, dθ, etc. (one time for each problem)

$\boxed{\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C}$ 10

(10) 3. $\int \frac{dt}{\sqrt{t^2+4}}$ [Hint: $\int \sec x dx = \ln|\sec x + \tan x| + C$]



$t = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dt = 2\sec^2\theta d\theta$

$\sqrt{t^2+4} = 2\sec\theta$ ⑤

$$\int \frac{dt}{\sqrt{t^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta} d\theta = \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{t^2+4}}{2} + \frac{t}{2}\right| + C$$

③ or ②

$$\boxed{\ln\left|\sqrt{t^2+4} + t\right| + C}$$

10

(10) 4. $\int \frac{2x}{(x-1)^2} dx$.

$$\frac{2x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$ ① ①

← Rule *

No additional points beyond this point if anything is wrong here

$2x = Ax - A + B$

$A = 2$

$B - A = 0 \rightarrow B = 2$

$$\int \frac{2x}{(x-1)^2} dx = \int \left[\frac{2}{x-1} + \frac{2}{(x-1)^2}\right] dx = 2\ln|x-1| - \frac{2}{x-1} + C$$ ② ②

or: $\int \frac{2x}{(x-1)^2} dx = \int \frac{2(u+1)}{u^2} du = 2\int\left(\frac{1}{u} + \frac{1}{u^2}\right) du$ ④ ②

$u = x-1 \rightarrow x = u+1$

$du = dx$

$= 2\left(\ln|u| - \frac{1}{u}\right) + C$

$= 2\ln|x-1| - \frac{2}{x-1} + C$

-1pt for missing abs. value

② ②

$$\boxed{2\ln|x-1| - \frac{2}{x-1} + C}$$

10

(10) 5. Find the area of the region under the curve $y = \frac{1}{x^3+x}$ from $x = 1$ to $x = 2$.

$$A = \int_1^2 \frac{1}{x^3+x} dx =$$ ②

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ ① ① Rule *

$1 = Ax^2 + A + Bx^2 + Cx$

$A + B = 0$

$C = 0$

$A = 1$

$B = -1$

$$= \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$ ② ②

$$= \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^2$$

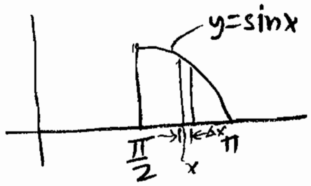
$$= \ln 2 - \frac{1}{2}\ln 5 + \frac{1}{2}\ln 2$$

or ②

$$\boxed{\frac{3}{2}\ln 2 - \frac{1}{2}\ln 5}$$

10

(10) 6. Find the volume of the solid of revolution obtained by rotating about the x–axis the region bounded by the curves $x = \frac{\pi}{2}$, $y = 0$ and $y = \sin x$, $\frac{\pi}{2} \le x \le \pi$.



$$V = \int_{\frac{\pi}{2}}^{\pi} \pi \sin^2 x \, dx \quad (1)\ (3)\ (1)\ (1)$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}(1-\cos 2x)\,dx \quad (2)$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{2}\left[\pi - \frac{\pi}{2}\right]$$

(2) $\boxed{\frac{\pi^2}{4}}$ $\boxed{10}$

(8) 7. Determine whether the integral below is convergent or divergent, and find its value if it is convergent. Important: You must use the definition of improper integrals.

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t x e^{-x^2}\,dx \quad (4)$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right] = \frac{1}{2} \quad (2)$$

-1pt for early omission of $\lim_{t\to\infty}$

(2) $\boxed{\frac{1}{2}}$ $\boxed{8}$

(8) 8. For each of the improper integrals below, circle C if it is convergent or D if it is divergent. (You need not show work for this problem).

NPC

(a) $\int_{2\pi}^{\infty} \sin\theta\, d\theta = \lim_{t\to\infty} \int_{2\pi}^{t} \sin\theta\, d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^{t}$ DNE C D (2)

(b) $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_2^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\left[-2\sqrt{3-x}\right]_2^t = 2$ C D (2)

(c) $\int_{-\infty}^{0} \frac{dx}{\sqrt{2-x}} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{\sqrt{2-x}} = \lim_{t\to-\infty}\left[-2\sqrt{2-x}\right]_t^0 = \infty$ C D (2)

(d) $\int_0^2 \frac{dx}{(x-1)^2}$ $\int_0^1 \frac{dx}{(x-1)^2} = \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^2} = \lim_{t\to 1^-}\left[-\frac{1}{x-1}\right]_0^t = \infty$ C D (2)

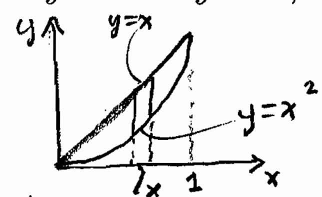
(8) 9. Find the length of the curve $y = \frac{2}{3}\, x^{3/2}$, $0 \le x \le 3$. ④

$$L = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^3 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \text{ ④}$$

$\boxed{\frac{14}{3}}$ $\boxed{8}$

(9) 10. Consider the lamina bounded by the curves $y = x$ and $y = x^2$, and with density $\rho = 1$. Find the following.



(a) The mass m of the lamina

$$m = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$\boxed{m = \frac{1}{6}}$ ③

(b) The moment M_y of the lamina about the y–axis.

$$M_y = \int_0^1 x(x - x^2)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$\boxed{M_y = \frac{1}{12}}$ ③

(c) The x–coordinate $\overline{x}$ of the center of mass of the lamina.

$$m\overline{x} = M_y$$

$$\frac{1}{6}\overline{x} = \frac{1}{12} \qquad \overline{x} = \frac{1}{2}$$

OK if correct consistently with above.

$\boxed{\overline{x} = \frac{1}{2}}$ ③

(8) 11. Determine whether each sequence below converges or diverges and if it converges find its limit. (You need not show work for this problem).

NPC

(a) $\{(-1)^n n e^{-n}\}$

$$\lim_{n\to\infty} |(-1)^n n e^{-n}| = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$\boxed{0}$ ②

(b) $a_n = \dfrac{\sin^2 n}{n+3}$

$$0 \le \sin^2 n \le 1 \to 0 \le \frac{\sin^2 n}{n+3} \le \frac{1}{n+3}$$

by squeeze theorem $0 \therefore \frac{\sin^2 n}{n+3} \to 0$, $\frac{1}{n+3} \to 0$ as $n\to\infty$

$\boxed{0}$ ②

(c) $\{\cos(n\pi)\}$

$-1, 1, -1, 1, -1, \ldots$

$\boxed{\text{diverges}}$ ②

(d) $a_n = n\sin(\frac{1}{n})$

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos(\frac{1}{x})(-\frac{1}{x^2})}{(-\frac{1}{x^2})} = 1$$

$\boxed{1}$ ②