NAME GRADING KEY

10–DIGIT PUID ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

Page 1	/15
Page 2	/29
Page 3	/32
Page 4	/24
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators nor any electronic devices may be used on this test.

Find the integrals in problems 1–6.

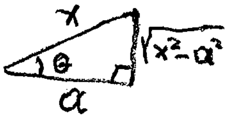
(6) 1. $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1+\cos 2x)\,dx = \frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$ ② ②

$= \frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{2}\cdot 1\right] = \frac{\pi+2}{8}$ ②

$$\boxed{\frac{\pi+2}{8}}$$

$\boxed{6}$

(9) 2. $\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}\,a\sec\theta\tan\theta\,d\theta =$ ④



$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

$= \frac{1}{a^2}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{a^2}\int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^3\theta}}\,d\theta$

$= \frac{1}{a^2}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{a^2}\,\frac{\sin^3\theta}{3} + C$ ③

$= \frac{1}{3a^2}\left(\frac{\sqrt{x^2-a^2}}{x}\right)^3 + C$

or ②

$$\boxed{\frac{1}{3a^2}\,\frac{(x^2-a^2)^{3/2}}{x^3} + C}$$

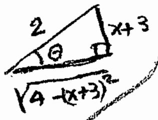
−1 pt for missing +C (one time for test)
−1pt for missing dx, du, dθ etc. (one time for each problem)

⑦

$$= \int_{\substack{u=x+3\\du=dx}} \frac{du}{\sqrt{4-u^2}} = \sin^{-1}\frac{u}{2} + C = \sin^{-1}\frac{x+3}{2} + C$$

Ⓞⓡ

(9) 3. $\displaystyle\int \frac{dx}{\sqrt{4-(x+3)^2}} =$



$x+3 = 2\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 2\cos\theta\, d\theta$

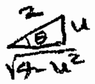
$\sqrt{4-(x+3)^2} = 2\cos\theta$

$$= \underbrace{\int \frac{2\cos\theta}{2\cos\theta} d\theta}_{④} = \int 1\, d\theta$$

$$= \underset{③}{\theta + C} = \sin^{-1}\frac{x+3}{2} + C$$

Ⓞⓡ $u = x+3,\ du = dx$

$$= \int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos\theta}{2\cos\theta} d\theta = \int 1 d\theta \quad ③$$



$u = 2\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = 2\cos\theta\, d\theta$

$\sqrt{4-u^2} = 2\cos\theta$

④ $= \theta + C$

$= \sin^{-1}\frac{u}{2} =$

②

$$\boxed{\sin^{-1}\frac{x+3}{2} + C}$$

$\boxed{9}$

(11) 4. $\displaystyle\int \frac{x^2+8x-3}{x^3+3x^2} dx$

$$② \rightarrow \frac{x^2+8x-3}{x^2(x+3)} = \overset{①}{\frac{A}{x}} + \overset{①}{\frac{B}{x^2}} + \overset{①}{\frac{C}{x+3}} \longleftarrow$$

Rule *

No additional points beyond this point if anything is wrong here

$x^2+8x-3 = Ax^2 + 3Ax + Bx + 3B + Cx^2$

$A + C = 1 \longrightarrow \boxed{C = -2}$

$3A + B = 8 \longrightarrow 3A - 1 = 8 \rightarrow \boxed{A = 3}$

$3B = -3 \longrightarrow \boxed{B = -1}$

$$\int \frac{x^2+8x-3}{x^2(x+3)} dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right) dx$$

$$= 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$$

Rule **

-1pt if abs. value is missing

$$\boxed{\overset{②}{3\ln|x|} + \overset{②}{\frac{1}{x}} - \overset{②}{2\ln|x+3|} + C}$$

$\boxed{11}$

(9) 5. $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x} dx = \int \frac{du}{u^2+u} =$ ②

$u = \sin x$

$du = \cos x\, dx$

$$① \rightarrow \frac{1}{u(u+1)} = \overset{①}{\frac{A}{u}} + \overset{①}{\frac{B}{u+1}}$$

Rule *

$1 = Au + A + Bu$

$A + B = 0$

$A = 1 \quad B = -1$

$$= \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \ln|\sin x| - \ln|\sin x + 1| + C$$

Rule **

$$\boxed{\overset{②}{\ln|\sin x|} - \overset{②}{\ln|\sin x + 1|} + C}$$

$\boxed{9}$

(6) 6. $\int \frac{x+1}{x^2+9}\,dx = \int\left(\frac{x}{x^2+9} + \frac{1}{x^2+9}\right)dx$

$= \frac{1}{2}\ln|x^2+9| + \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

③ ③

$$\boxed{\frac{1}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3} + C}$$

$\boxed{6}$

(10) 7. Find the area of the region under the curve $y = \frac{x+1}{x-1}$ from $x = 2$ to $x = 3$.

$A = \underbrace{\int_2^3 \frac{x+1}{x-1}\,dx}_{④} = \int_2^3 \left(1 \overset{③}{+} \frac{2}{x-1}\right)dx = \left[x + 2\ln|x-1|\right]_2^3$

$= 3 + 2\ln 2 - 2$ ③

$\left(x-1 \overset{1}{\overline{)\,x+1}} \quad \frac{x-1}{2}\right) \rightarrow \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$

or $\frac{x+1}{x-1} = \frac{x-1+2}{x-1} = 1 + \frac{2}{x-1}$

$$\boxed{1 + 2\ln 2}$$

$\boxed{10}$

(10) 8. Determine whether the integral below is convergent or divergent. Find its value if it is convergent. Important: You must use the definition of improper integrals.

$\int_{-\infty}^{0} \frac{1}{2x-5}\,dx = \overbrace{\lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx}^{④} = \lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0$

$= \lim_{t\to-\infty}\left[\underset{②}{\frac{1}{2}\ln 5} - \underset{②}{\frac{1}{2}\ln|2t-5|}\right]$

$= \frac{1}{2}\ln 5 - \infty = -\infty$

-1 pt for early omission of $\lim_{t\to-\infty}$

②

$\boxed{\text{divergent}}$

$\boxed{10}$

(6) 9. Circle T if true or F if false.

(a) $\int_1^\infty \frac{1}{x\sqrt{x}}\,dx$ is divergent ↖ $p = \frac{3}{2}$ $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$. T [✓ F] ②

(b) $\int_{-1}^{2} \frac{1}{x^2}\,dx = -\frac{3}{2}$ The fundamental theorem of Calculus does not apply because the integrand $f(x) = \frac{1}{x^2}$ is not continuous at $x=0$. T [✓ F] ②

(c) $\overbrace{\int_0^1 \frac{1}{\sqrt{1-x}}\,dx \text{ is convergent}}$ $= \lim_{t\to1^-}\int_0^t \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to1^-}\left[-2\sqrt{1-x}\right]_0^t$ $= \lim_{t\to1^-}\left[-2\sqrt{1-t} + 2\right] = 2$ [✓ T] F ②

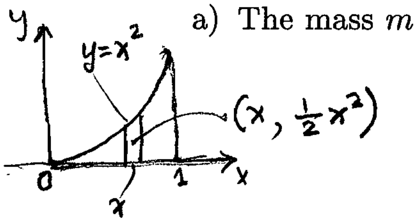
(5) 10. Set up but do not evaluate an integral for the length L of the curve $y = \tan x$, $0 \le x \le 1$.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{L = \int_0^1 \sqrt{1+\sec^4 x}\;dx}$$

$\boxed{5}$

(9) 11. Consider the lamina bounded by the curves $y = x^2$, $y = 0$, $x = 1$ and with density $\rho = 1$. Find the following:



a) The mass m of the lamina.

$$m = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\boxed{m = \frac{1}{3}}$$

③

b) The moment M_x of the lamina about the x–axis.

$$M_x = \int_0^1 \frac{x^2}{2} x^2 dx = \frac{x^5}{10}\Big|_0^1 = \frac{1}{10}$$

$$\boxed{M_x = \frac{1}{10}}$$

③

c) The y–coordinate of the center of mass of the lamina.

$$m\bar{y} = M_x$$

$$\frac{1}{3}\bar{y} = \frac{1}{10}$$

Ok if consistent with above

$$\boxed{\bar{y} = \frac{3}{10}}$$

③

(10) 12. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this porblem).

NPC

(a) $\left\{\tan\frac{\pi}{3n}\right\}$

$\frac{\pi}{3n} \to 0$ as $n \to \infty$

$\therefore \tan\frac{\pi}{3n} \to 0$ as $n \to \infty$

$\boxed{0}$ ②

(b) $a_n = \sin\frac{n\pi}{2}$

0, 1, 0, −1, 0, 1....

$\boxed{\text{diverges}}$ ②

(c) $\left\{n^2 e^{-n}\right\}$

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \underset{\frac{\infty}{\infty}}{\frac{x^2}{e^x}} \overset{L'H}{=} \lim_{x\to\infty} \underset{\frac{\infty}{\infty}}{\frac{2x}{e^x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$$\therefore \lim_{n\to\infty} n^2 e^{-n} = 0$$

$\boxed{0}$ ②

(d) $a_n = \dfrac{(-1)^{n-1}n}{n^2+1}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

$$\therefore \lim_{n\to\infty} a_n = 0$$

$\boxed{0}$ ②

(e) $a_n = \dfrac{3n^2+5}{n^2+2n-1}$

$$\lim_{n\to\infty} \frac{3n^2+5}{n^2+2n-1} = \lim_{n\to\infty} \frac{3+\frac{5}{n^2}}{1+\frac{2}{n}-\frac{1}{n}} = 3$$

$\boxed{3}$ ②