NAME GRADING KEY

STUDENT ID ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

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TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Evaluate the integrals in problems 1–5.

(6) 1. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$ ③

$= \tan x - x + C$ ③

$\boxed{\tan x - x + C}$ 6

(12) 2. $\int \frac{x+1}{x^3+x}\, dx$

$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ ② ② ②

No additional credit beyond this point if anything is wrong here.

$x+1 = Ax^2 + A + Bx^2 + Cx$

$A+B=0 \quad C=1 \quad A=1 \quad B=-1$

$\int \frac{x+1}{x^3+x}\, dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+1}\right) dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx$

$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + C$

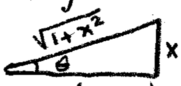
−1 pt for missing +C (one time only for test)

−1 pt for missing dx, du, etc (one time only for each problem)

② ② ②

$\boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + C}$ 12

(12) 3. $\int x^3\sqrt{1+x^2}\,dx$ $= \int \tan^3\theta \sec^3\theta\, d\theta =$ 6 ← 0 pts for problem if this wrong



$x = \tan\theta \quad dx = \sec^2\theta\, d\theta$

$\sqrt{1+x^2} = \sec\theta$

Note The substitution $u = 1+x^2$ will also work for this problem

$= \int \tan^2\theta \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta$

$= \int (\sec^2\theta - 1)\sec^2\theta \sec\theta\tan\theta\, d\theta \quad = \int (u^4 - u^2)\,du$

$u = \sec\theta \quad du = \sec\theta\tan\theta\, d\theta$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + C$

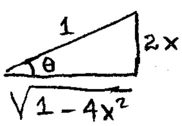
$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$

3 pts if answer is left in terms of θ and is correct.

3 3

$\boxed{\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C}$ 12

(12) 4. $\int \sqrt{1-4x^2}\,dx$ $= \frac{1}{2}\int \cos^2\theta\, d\theta$ 6



$2x = \sin\theta \quad dx = \frac{1}{2}\cos\theta\, d\theta$

$\sqrt{1-4x^2} = \cos\theta$

Same grading rules as in problem 3.

$= \frac{1}{4}\int (1+\cos 2\theta)\,d\theta = \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$= \frac{1}{4}\left(\theta + \sin\theta\cos\theta\right) + C$

$= \frac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C$

3 pts if answer is left in terms of θ and is correct

or → 3 3 $\boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$ 12

(6) 5. $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$ $= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x\,dx =$ 3

$= \left[\sin x - \frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$

3

$\boxed{\frac{2}{3}}$ 6

-1 pt each time "=" and "lim" are inconsistent, (at most -2pts for this)

(12) 6. Determine whether each integral is convergent or divergent. Find its value if it is convergent. Important: Show clearly how limits are involved.

(a) $\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t xe^{-x^2}\,dx$ 3

$= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_0^t = \overbrace{\lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right)}^{2} = \frac{1}{2}$ 1

$\boxed{\frac{1}{2}}$ 6

(b) $\int_1^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx$ 3

$= \lim_{t\to 1^+} \left[\ln|x-1|\right]_t^3 = \overbrace{\lim_{t\to 1^+}\left[\ln 2 - \ln|t-1|\right]}^{2} = \infty$

-1pt for $-\infty$

$\boxed{\text{divergent}}$ 1 6

(10) 7. Find the length of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$, $0 \le x \le 1$.

$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ 4

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{2}(x^2+2)^{1/2}2x\right)^2 = 1 + x^2(x^2+2)$

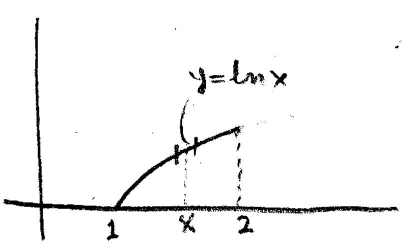
$= 1 + x^4 + 2x^2 = (1+x^2)^2$

$\therefore L = \int_0^1 \sqrt{(1+x^2)^2}\,dx = \int_0^1 (1+x^2)\,dx$ 5

$= \left(x + \frac{x^3}{3}\right)\Big|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}$ 1

$\boxed{\frac{4}{3}}$ 10

(8) 8. Set up an integral for the area ~~A~~ S of the surface obtained by rotating the curve $y = \ln x$, $1 \le x \le 2$, about the x-axis. Do not evaluate the integral.



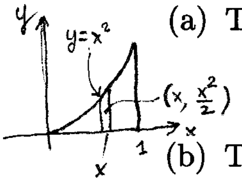
$dS = 2\pi y\,ds$

$\boxed{\cancel{A}\,S = \int_1^2 \overbrace{2\pi \ln x \sqrt{1 + \frac{1}{x^2}}}^{5}\,dx}$ 1 1 1

8

(12) 9. Consider the lamina bounded by the curves $y = x^2$, $y = 0$, $x = 1$, and with density $\rho = 1$. Find the following:

(a) The mass m of the lamina.



$m = \int_0^1 1\, x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$\boxed{m = \frac{1}{3}}$ 2 NPC

(b) The moment M_y of the lamina about the y-axis.

$M_y = \int_0^1 x\, x^2 dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$

$\boxed{M_y = \frac{1}{4}}$ 3 NPC

(c) The moment M_x of the lamina about the x-axis.

$M_x = \int_0^1 \frac{x^2}{2} x^2 dx = \frac{x^5}{10}\Big|_0^1 = \frac{1}{10}$

$\boxed{M_x = \frac{1}{10}}$ 3 NPC

(d) The center of mass $(\bar{x}, \bar{y})$ of the lamina.

$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$, $\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10}$

2 2

$\boxed{(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{3}{10}\right)}$ OK if consistent with above

~~(24)~~ (10) 10. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem).

2 pts each NPC

(a) $a_n = \cos n\pi$

$-1, 1, -1, 1, -1, \ldots$

$\boxed{\text{diverges}}$ 2

(b) $a_n = \dfrac{\sqrt{n}}{1+n}$

$\lim_{x\to\infty} \frac{\sqrt{x}}{1+x} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{x}+1} = 0$

$\boxed{0}$ 2

(c) $a_n = \dfrac{\ln n^2}{n}$

$\lim_{x\to\infty} \underbrace{\frac{\ln x^2}{x}}_{\frac{\infty}{\infty}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0$

$\boxed{0}$ 2

(d) $a_n = \dfrac{n \sin n}{n^2+1}$ $\quad -\frac{n}{n^2+1} \le \frac{n \sin n}{n^2+1} \le \frac{n}{n^2+1}$

squeeze theorem

as $n\to\infty$: $\downarrow 0$ $\quad \therefore \downarrow 0$ $\quad \downarrow 0$

$\boxed{0}$ 2

(e) $a_n = \dfrac{3n^2 - 2n + 1}{2n^2 + n - 1}$

$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{2x^2 + x - 1} = \frac{3}{2}$

$\boxed{\frac{3}{2}}$ 2