NAME SOLUTIONS

STUDENT ID # ______________________

RECITATION INSTRUCTOR ______________________

RECITATION TIME ______________________

LECTURER ______________________

INSTRUCTIONS

1. There are 8 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. ID # is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–8.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 9 digit student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. $\lim_{x\to 0} x^2 \cos \frac{\pi}{x} =$

$-1 \le \cos\frac{\pi}{x} \le 1$

$-x^2 \le \underbrace{x^2\cos\frac{\pi}{x}}\le x^2$ since $x^2 > 0$

as $x \to 0$ $\quad \downarrow \qquad \therefore \downarrow \qquad \downarrow$

$\quad 0 \qquad 0 \qquad 0$

by the Squeeze theorem

A. π

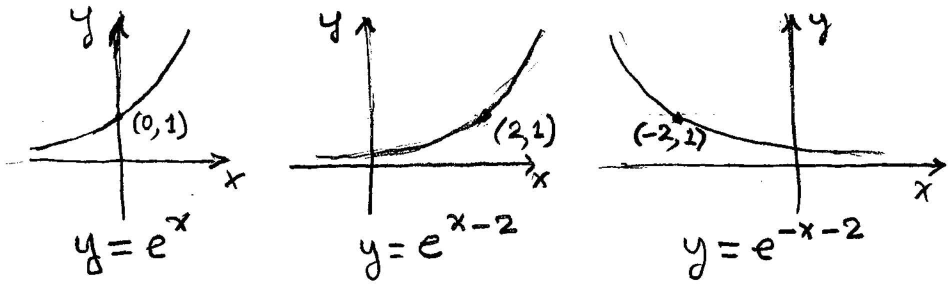
B. 0

C. -1

D. ∞

E. does not exist

2. Starting with the graph of $y = e^x$, the equation of the graph that results by shifting 2 units to the right and then reflecting about the y–axis is



A. $y = -e^{x-2}$

B. $y = -e^{-x+2}$

C. $y = e^{-x-2}$

D. $y = e^{x+2}$

E. $y = e^{x-2}$

3. Find the value of the constant c for which the following function g is continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \ge 4 \end{cases}$$

g is continuous on $(-\infty, 4)$ and on $(4, \infty)$ because it is a polynomial in each interval.

g cont. at $x = 4$: $\lim_{x\to 4} g(x) = g(4)$

$\lim_{x\to 4} g(x)$ exists $\Leftrightarrow$ $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x)$

$16 - c^2 = c4 + 20$

or $c^2 + 4c + 4 = 0$

$(c+2)^2 = 0 \longrightarrow c = -2$

When $c = -2$: $\lim_{x\to 4} g(x) = 12 = g(4)$

A. 2

B. -2

C. 4

D. -4

E. for no value of c

4. $\lim_{h\to 0} \frac{\sin(3+h)^2 - \sin 9}{h} =$ $f'(3)$ where $f(x) = \sin x^2$ by definition of $f'(a)$

$f'(x) = 2x\cos x^2$

$f'(3) = 2\cdot 3\cos 3^2 = 6\cos 9$

You can also use L'Hospital's rule

A. 0

B. ∞

C. $6 \sin 9$

D. $6 \cos 9$

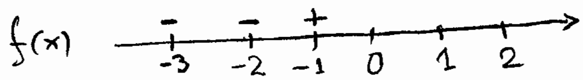
E. does not exist

5. The equation $x^3 + 1 = x$ has exactly one root in the interval $(-3, 2)$. This root is in the interval

$x^3 - x + 1 = 0$

Let $f(x) = x^3 - x + 1$

$f(-3) = -24 < 0$
$f(-2) = -5 < 0$
$f(-1) = 1 > 0$



The root is in $(-2,-1)$ by the Intermediate value theorem

A. $(-3, -2)$
B. $(-2, -1)$
C. $(-1, 0)$
D. $(0, 1)$
E. $(1, 2)$

6. $\lim_{x\to\infty} \tan^{-1}(x^2 - x^4) =$

Let $u = x^2 - x^4$

$\lim_{x\to\infty} u = \lim_{x\to\infty} (x^2 - x^4) = \lim_{x\to\infty} -x^4\left(1 - \frac{1}{x^2}\right) = -\infty$

$\therefore \lim_{x\to\infty} \tan^{-1}(x^2 - x^4) = \lim_{u\to-\infty} \tan^{-1} u = -\frac{\pi}{2}$

A. $\frac{\pi}{3}$
B. $-\frac{\pi}{3}$
C. ∞
D. $-\infty$
E. $-\frac{\pi}{2}$

7. If $y = \tan^2(x^2)$, then $\frac{dy}{dx} =$ $2\tan(x^2)\sec^2(x^2) \cdot 2x$

$= 4x \tan(x^2)\sec^2(x^2)$

A. $2x\sec^2(x^2)$
B. $4x\tan(x^2)\sec^2(x^2)$
C. $4x\tan^2(x^2)\sec^2(x^2)$
D. $4x\tan(x^2)\sec(x^2)$
E. $2\tan^2(x^2)\sec(x^2)$

8. If $y = \tan^{-1}(x^2)$, then $\frac{d^2y}{dx^2} =$

$$\frac{dy}{dx} = \frac{2x}{1+x^4}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^4)2 - 2x\,4x^3}{(1+x^4)^2}$$

$$= \frac{2+2x^4-8x^4}{(1+x^4)^2} = \frac{2-6x^4}{(1+x^4)^2}$$

A. $\frac{2x}{1+x^4}$
B. $\frac{6x^4-2}{(1+x^4)^2}$
C. $\frac{-4}{(1+x^4)^2}$
D. $\frac{2-6x^4}{(1+x^4)^2}$
E. $\frac{8x^4}{(1+x^4)^2}$

9. Find the slope of the tangent line to the curve $\cos x \sin y = \frac{1}{2\sqrt{2}}$ at the point $\left(\frac{\pi}{3}, \frac{\pi}{4}\right)$.

$$\cos x \sin y = \frac{1}{2\sqrt{2}}$$

$$\cos x \cos y \cdot \frac{dy}{dx} - \sin x \sin y = 0$$

When $(x,y) = \left(\frac{\pi}{3}, \frac{\pi}{4}\right)$: $\cos\frac{\pi}{3} \cos\frac{\pi}{4} \frac{dy}{dx} - \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 0$

$$\frac{1}{2}\frac{1}{\sqrt{2}}\frac{dy}{dx} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = 0$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{3}, \frac{\pi}{4}\right)} = \sqrt{3}$$

A. $\frac{1}{\sqrt{3}}$

B. $\sqrt{3}$

C. $-\sqrt{3}$

D. $-\frac{1}{\sqrt{3}}$

E. $\frac{2}{\sqrt{3}}$

10. Find an equation of the tangent line to the curve $y = 4^x$ at the point where $x = 2$.

$$y = 4^x = e^{\ln 4^x} = e^{x \ln 4}$$

$$\frac{dy}{dx} = e^{x \ln 4} \ln 4 = (\ln 4)\, 4^x$$

$$\left.\frac{dy}{dx}\right|_{x=2} = (\ln 4)\, 4^2 = 16 \ln 4$$

when $x = 2$: $y = 4^2 = 16$

eq. of tan. line $y - 16 = 16(\ln 4)(x-2)$

A. $y - 8 = \frac{16}{\ln 4}(x-2)$

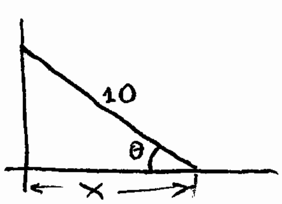
B. $y - 2 = 8(x - 16)$

C. $y - 16 = 16e^4(x-2)$

D. $y - 2 = (\ln 4)(x - 16)$

E. $y - 16 = 16(\ln 4)(x-2)$

11. A ladder 10 ft long is leaning against a wall. The bottom of the ladder is slipping away from the wall causing the acute angle between the ladder and the ground to decrease at the rate of 2 rads/sec. How fast is the bottom of the ladder moving when the acute angle between the ladder and the ground is $\frac{\pi}{6}$ rads?



$$\frac{d\theta}{dt} = -2 \text{ rads/sec}$$

Find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{6}$ rads

$$x = 10 \cos\theta$$

$$\frac{dx}{dt} = -10 \sin\theta \frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{6}$: $\frac{dx}{dt} = -10 \sin\frac{\pi}{6}(-2) = 10$

A. 10 ft/sec

B. 5 ft/sec

C. $10\sqrt{3}$ ft/sec

D. $\frac{10}{\sqrt{3}}$ ft/sec

E. 2 ft/sec

12. If $f(x) = (\sin^{-1} x)^2$, then $f'(x) =$ $2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$

A. $-2(\sin x)^{-3}\cos x$

B. $2\sin^{-1} x \cos^{-1} x$

C. $-2\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

D. $\frac{2\sin^{-1} x}{\sqrt{1-x^2}}$

E. $\frac{2\sin^{-1} x}{1+x^2}$

$$f(x) = \begin{cases} 3-(x-2) = 5-x, & \text{if } x \geq 2 \\ 3-[-(x-2)] = 1+x, & \text{if } x<2 \end{cases}$$

13. The minimum and maximum values of the function $f(x) = 3 - |x-2|$ on the interval $[1,4]$ are

The only critical number is $x=2$ where the derivative of f does not exist

$f(1) = 3-|1-2| = 2$

$f(2) = 3$ ← max

$f(4) = 3-(4-2) = 1$ ← min

A. 1 and 3

B. 0 and 1

C. 2 and 3

D. 2 and 4

E. 0 and 2

14. The derivative of the function f is given by $f'(x) = (x-1)x(x+1)$. The function f has a local minimum at $x =$

$f'(x)$: $- - - 0 + + + + + 0 - - - - 0 + + + +$ at $x = -1,\ 0,\ 1$

-1 ↑ min, 1 ↑ min

because the derivative changes from negative to positive

A. 0

B. $-\frac{1}{\sqrt{3}}$

C. $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$

D. -1 and 1

E. $0, -1$, and 1

15. The graph of the function $f(x) = 3x^5 - 20x^4 + 1000$ has an inflection point when $x =$

$f'(x) = 15x^4 - 80x^3$

$f''(x) = 60x^3 - 240x^2 = 60x^2(x-4)$

$f''(x)$: $- - - - 0 - - - - 0 + + + +$ at $x = 0,\ 4$

C.D. 0 C.D. 4 C.U.

↑ inflection point when $x=4$ because concavity changes from Down to Up.

A. 0

B. 0 and $\frac{16}{3}$

C. 0 and 4

D. 4

E. $\frac{16}{3}$ and 4

16. $\lim\limits_{x\to 0^+} \dfrac{\ln(1+x)}{\ln(1-x^2)} \overset{L'H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{1+x}}{\frac{1}{1-x^2}(-2x)}$

$\frac{0}{0}$

$= \lim\limits_{x\to 0^+} \dfrac{1-x^2}{(1+x)(-2x)}$

$= \lim\limits_{x\to 0^+} \dfrac{(1+x)(1-x)}{(1+x)(-2x)}$

$= \lim\limits_{x\to 0^+} \dfrac{1-x}{-2x} = -\infty$

A. 0

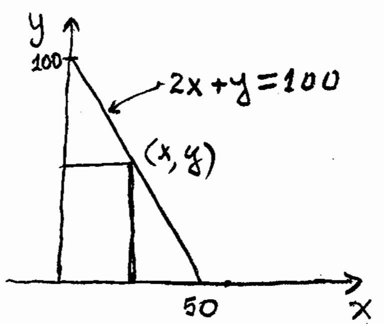
B. 1

C. e

D. $-\infty$

E. ∞

17. Find the area of the largest rectangle with vertices at the origin, a point on the positive x–axis, a point on the positive y–axis and a point on the line $2x + y = 100$.



$A = xy$

$A = x(100-2x)$

$A = 100x - 2x^2, \ 0 \le x \le 50$

$\dfrac{dA}{dx} = 100 - 4x$

$\dfrac{dA}{dx} = 0 : \ 100 - 4x = 0 \rightarrow x = 25$

$\dfrac{dA}{dx}$: +++0--- (0, 25, 50, x) $\therefore$ A is max at $x=25$

max $A = 100\cdot 25 - 2\cdot 625 = 2500 - 1250 = 1250$

A. 500

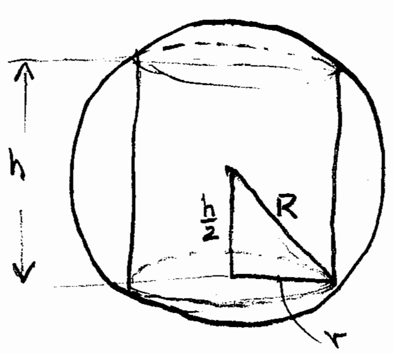
B. 1750

C. 1000

D. 5000

E. 1250

18. Find the **height** of a right circular cylinder of largest volume that can be inscribed in a sphere of radius R.



$V = \pi r^2 h$

$r^2 + \left(\frac{h}{2}\right)^2 = R^2$

$r^2 = R^2 - \frac{h^2}{4}$

$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$V = \pi R^2 h - \frac{\pi}{4}h^3, \ 0 \le h \le 2R$

$\dfrac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2$

$\dfrac{dV}{dx} = 0 : \ \pi R^2 - \frac{3}{4}\pi h^2 = 0 \rightarrow h = \frac{2R}{\sqrt{3}}$ ← V is max because $V=0$ at $h=0, 2R$

A. $\dfrac{R}{\sqrt{2}}$

B. $\dfrac{2R}{\sqrt{3}}$

C. $\dfrac{R}{3}$

D. $\sqrt{3}\,R$

E. $\dfrac{\pi}{2}\,R$

19. If $f'(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}$ and $f(4) = 3$, then $f(x) =$

$f'(x) = 3x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$f(x) = 3\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}} + C$

$f(x) = 2x^{3/2} - 2x^{1/2} + C$

$x=4$: $3 = 2\cdot 4^{3/2} - 2\cdot 4^{1/2} + C$

$3 = 16 - 4 + C \to C = -9 \quad \therefore f(x) = 2x^{3/2} - 2\sqrt{x} - 9$

A. $2x^{3/2} - 2\sqrt{x} - 9$

B. $x^{3/2} - 2\sqrt{x} - 1$

C. $2x^{3/2} + \sqrt{x} - 15$

D. $2x^{3/2} - 2\sqrt{x} + 3$

E. $2x^{3/2} - \sqrt{x} - 11$

20. $\int_{-e^2}^{-e} \frac{3}{x}\, dx = 3\ln|x| \Big|_{-e^2}^{-e}$

$= 3\ln e - 3\ln e^2$

$= 3\cdot 1 - 3\cdot 2\cdot 1 = -3$

A. $e^3 - e^6$

B. -3

C. 3

D. $\frac{3}{e^4} - \frac{3}{e^2}$

E. $e^6 - e^3$

21. $\int_{3/8}^{1} \frac{1}{x^2}\sqrt{1 + \frac{3}{x}}\, dx = *$

Let $u = 1 + \frac{3}{x}$ $\quad du = -\frac{3}{x^2}dx \to \frac{1}{x^2}dx = -\frac{1}{3}du$

$x = \frac{3}{8} \to u = 9$

$x = 1 \to u = 4$

$\therefore * = \int_9^4 \left(-\frac{1}{3}\right)\sqrt{u}\, du = -\frac{1}{3}\frac{u^{3/2}}{\frac{3}{2}}\Big|_9^4$

$= -\frac{2u^{3/2}}{9}\Big|_9^4 = -\frac{2\cdot 4^{3/2}}{9} - \left(-\frac{2\cdot 9^{3/2}}{9}\right)$

$= -\frac{16}{9} + 6 = \frac{54-16}{9} = \frac{38}{9}$

A. $\frac{32\sqrt{2} - 3\sqrt{3}}{48\sqrt{2}}$

B. $\frac{32\sqrt{2} - 3\sqrt{3}}{16\sqrt{2}}$

C. $-\frac{38}{9}$

D. $\frac{38}{3}$

E. $\frac{38}{9}$

22. A year ago there were 4 grams of a certain radioactive substance. Today there are 3 grams. The half–life of the substance is

$m = m_0 e^{kt} \qquad m = 4e^{kt}$

$t=1$: $3 = 4e^{k1}$

$\frac{3}{4} = e^k \to k = \ln\frac{3}{4} = \ln 3 - \ln 4$

Find t for wich $m = \frac{1}{2}m_0$

$\frac{1}{2}m_0 = m_0 e^{(\ln 3 - \ln 4)t}$

$\frac{1}{2} = e^{(\ln 3 - \ln 4)t}$

$-\ln 2 = (\ln 3 - \ln 4)t \to t = \frac{\ln 2}{\ln 4 - \ln 3}$

A. $\frac{\ln 2}{\ln 3 - \ln 4}$ years

B. $\frac{\ln 2}{\ln 4 - \ln 3}$ years

C. $\frac{\ln 2}{3\ln 4}$ years

D. $\frac{\ln 2}{4\ln 3}$ years

E. $-\frac{\ln 2}{3\ln 4}$ years

23. $\dfrac{d}{dx}\displaystyle\int_0^{\sin x} e^{-t^2}\,dt =$ $e^{-\sin^2 x}\cos x$

A. $e^{-\sin^2 x}\cos x$

B. $e^{-\sin^2 x}$

C. $-2xe^{-x^2}\cos x$

D. $e^{-\sin^2 x}\sin x$

E. $-2xe^{-x^2}$

24. Find the vertex, focus and directrix of the parabola $(x-4)^2 = 8(y+3)$.

A. vertex $(4,-1)$, focus $(4,-3)$, directrix $y=-5$

B. vertex $(4,-3)$, focus $(4,-1)$, directrix $y=-5$

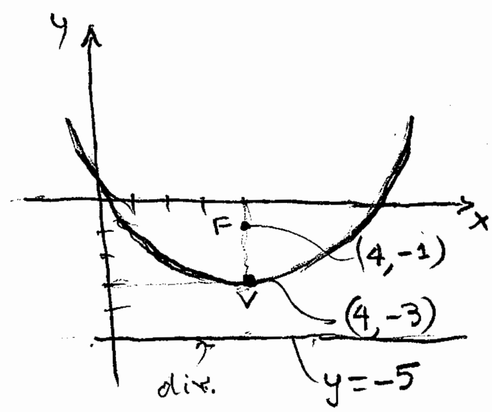
C. vertex $(-4,3)$, focus $(-4,5)$, directrix $y=1$

D. vertex $(-4,1)$, focus $(-4,3)$, directrix $y=-1$

E. vertex $(4,-3)$, focus $(4,-5)$, directrix $y=-1$

vertex $(4,-3)$

$4p = 8 \to p = 2$



25. Find the vertices and foci of the hyperbola $\dfrac{(y-2)^2}{4} - \dfrac{(x+3)^2}{3} = 1$.

A. vertices $(-3,0)$, $(-3,5)$; foci $(-3,4)$, $(-3,-1)$

B. vertices $(-3,4)$, $(-3,0)$; foci $(-3,5)$, $(-3,-1)$

C. vertices $(-3,4)$, $(-3,0)$; foci $(-3,2+\sqrt{7})$, $(-3,2-\sqrt{7})$

D. vertices $(-1,2)$, $(-5,2)$; foci $(-3-\sqrt{7},2)$, $(-3+\sqrt{7},2)$

E. vertices $(-1,2)$, $(-5,2)$; foci $(0,2)$, $(-6,2)$

center $(-3, 2)$

$a = 2 \quad b = \sqrt{3}$

$c^2 = a^2 + b^2 = 4 + 3$

$c = \sqrt{7}$