Answer Keys for Exam 2 Version 01.

1. $$F(\theta) = \sin^{-1}(\sqrt{\sin\theta})$$

Set
$$\begin{cases} u = \sin\theta \\ v = \sqrt{u} \\ w = \sin^{-1} v \end{cases}$$

Then

$$F'(\theta) = \frac{dF}{d\theta} = \frac{dw}{d\theta}$$
$$= \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{d\theta}.$$
$$= \frac{1}{\sqrt{1-v^2}} \cdot \frac{1}{2\sqrt{u}} \cdot \cos\theta$$
$$= \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta$$
$$= \frac{\cos\theta}{2\sqrt{1-\sin\theta}\cdot\sqrt{\sin\theta}}$$

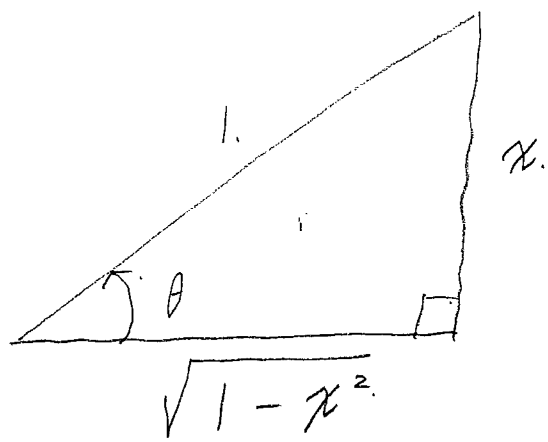
$$\boxed{\text{Answer A. } \frac{\cos\theta}{2\sqrt{1-\sin\theta}\sqrt{\sin\theta}}}$$

2. $\tan(\sin^{-1} x)$

Set $\sin^{-1} x = \theta$.

Then $x = \sin\theta$

Consider the following triangle



Then we have

$$\tan(\sin^{-1} x) = \tan\theta$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$\boxed{\text{Answer} \quad C : \frac{x}{\sqrt{1-x^2}}}$$

3. $f(x) + x^2 [f(x)]^3 = 10.$

Then

$$f'(x) + 2x [f(x)]^3$$
$$+ x^2 \cdot 3 [f(x)]^2 \cdot f'(x) = 0$$

Therefore,

$$f'(1) + 2 \cdot 1 \cdot [f(1)]^3$$
$$+ 1^2 \cdot 3 [f(1)]^2 \cdot f'(1) = 0$$

Since $f(1) = 2$, we have

$$f'(1) + 2 \cdot 1 \cdot 2^3$$
$$+ 1^2 \cdot 3 \cdot 2^2 \cdot f'(1) = 0$$

i.e.

$$13 f'(1) + 16 = 0.$$

$$\therefore \quad f'(1) = -\frac{16}{13}.$$

$$\boxed{\text{Answer} \quad A. \quad -\frac{16}{13}.}$$

4.

$$y \sin(2x) = x \cos(2y)$$

Apply $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{dy}{dx} \cdot \sin(2x) + y \cdot \cos(2x) \cdot 2$$

$$= 1 \cdot \cos(2y) + x \cdot \{-\sin(2y)\} 2 \cdot \frac{dy}{dx}$$

Plug in $(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ to the above equation to obtain

$$\cancel{\frac{dy}{dx} \sin(\pi)} + \frac{\pi}{4} \cos(\pi) \cdot 2$$

$$= 1 \cdot \cancel{\cos\left(\frac{\pi}{2}\right)} + \frac{\pi}{2} \left\{-\sin\left(\frac{\pi}{2}\right)\right\} 2 \cdot \frac{dy}{dx}$$

$$-\frac{\pi}{2} = -\pi \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

$$\boxed{\text{Answer} \quad A \quad \frac{1}{2}}$$

5.

$$y = x^{\ln x}$$

$$\ln y = \ln \left(x^{\ln x} \right)$$

$$= \ln x \cdot \ln x$$

$$= (\ln x)^2$$

$$\frac{1}{y} \frac{dy}{dx} = 2 \ln x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \cdot 2 \ln x \cdot \frac{1}{x}$$

$$= x^{\ln x} \cdot 2 \ln x \cdot \frac{1}{x}$$

$$\therefore \left. \frac{dy}{dx} \right|_{x=e} = e^{\ln e} \cdot 2 \ln e \cdot \frac{1}{e}$$

$$= e^1 \cdot 2 \cdot 1 \cdot \frac{1}{e} = 2$$

$\boxed{\text{Answer } \beta \quad 2.}$

6:

The linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 1000$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[3]{1000} + \frac{1}{3\cdot(\sqrt[3]{1000})^2}(x-1000)$$

$$= 10 + \frac{1}{300}(x-1000)$$

Therefore,

$$\sqrt[3]{1001} = f(1001)$$

$$\approx L(1001)$$

$$= 10 + \frac{1}{300}(1001-1000)$$

$$= \frac{3001}{300}$$

$$\boxed{\text{Answer} \quad E \quad \frac{3001}{300}}$$

7.

$$\{\ln(2\cosh x)\}'$$

$$= \frac{(2\cosh x)'}{2\cosh x}$$

$$= \frac{\cancel{2}\sinh x}{\cancel{2}\cosh x} = \tanh x$$

$$\boxed{\text{Answer} \quad E. \quad \tanh x}$$

8. $y = x^3 - 2x^2 + 1.$

$$dy = \frac{dy}{dx}\, dx$$

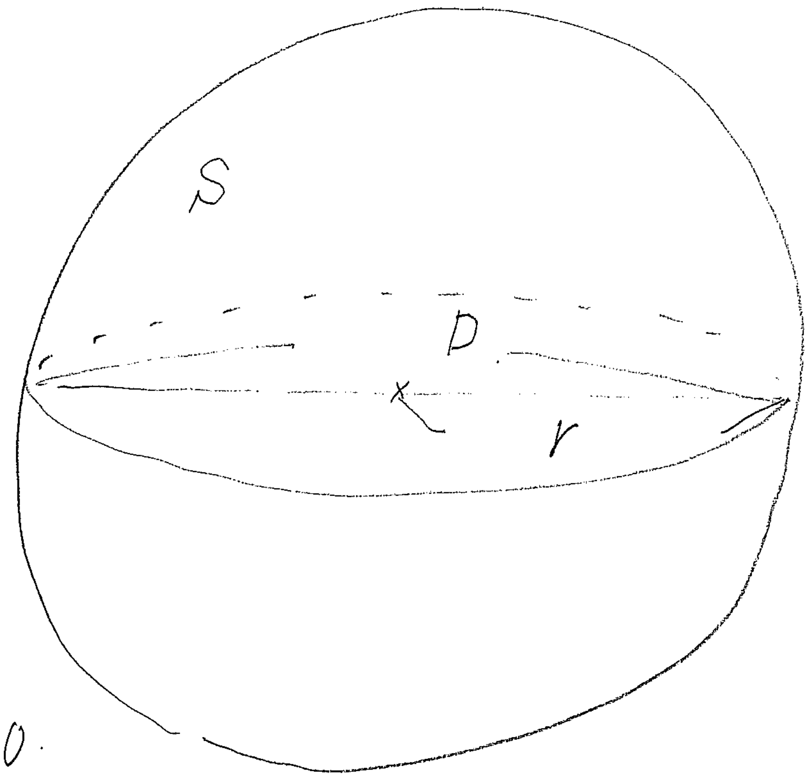
$$= (3x^2 - 4x)\, dx$$

$$= (3 \cdot 2^2 - 4 \cdot 2) \cdot 0.2$$

$$= 4 \cdot 0.2 = 0.8$$

$\boxed{\text{Answer B. } 0.8}$

9.



Given

$$\frac{dS}{dt} = -1.$$

Unknown.

$$\frac{dD}{dt} = ?$$

when $D = 10$.

Relation.

$$S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2.$$

$$= \pi D^2.$$

Solution

$$\frac{dS}{dt} = \pi \cdot 2D \frac{dD}{dt}$$

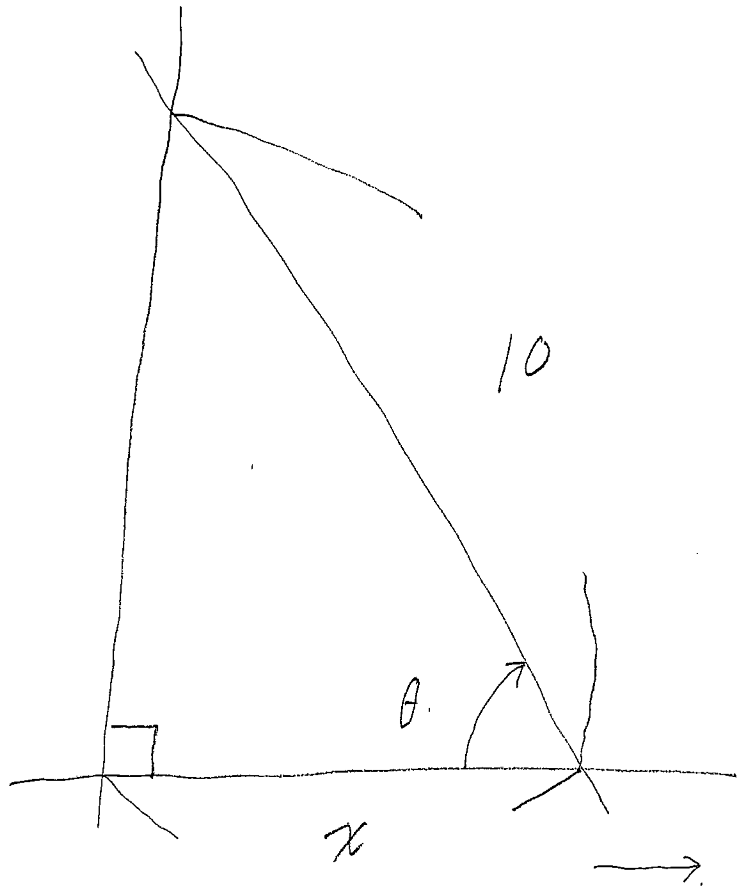
$$-1 = \pi \cdot 2 \cdot 10 \cdot \frac{dD}{dt}$$

$$\therefore \frac{dD}{dt} = -\frac{1}{20\pi}.$$

The diameter is decreasing at a rate of $\frac{1}{20\pi}$ cm/min.

$\boxed{\text{Answer } D \quad \frac{1}{20\pi} \text{ cm/min}}$

10



Given

$$\frac{dx}{dt} = 1.$$

Unknown.

$$\frac{d\theta}{dt} = ? \quad \text{when} \quad x = 6.$$

Relation

$$\frac{x}{10} = \cos\theta.$$

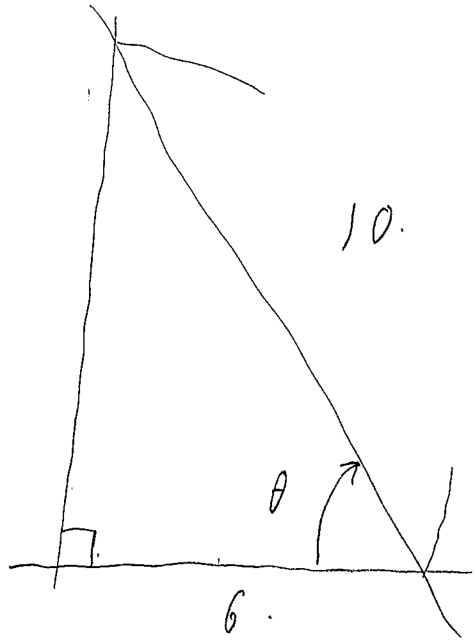
i.e.

$$x = 10\cos\theta.$$

Solution

$$\frac{dx}{dt} = -10 \sin\theta \cdot \frac{d\theta}{dt}$$

$$\sqrt{10^2 - 6^2} = 8$$



$$= -10 \cdot \frac{8}{10} \frac{d\theta}{dt}$$

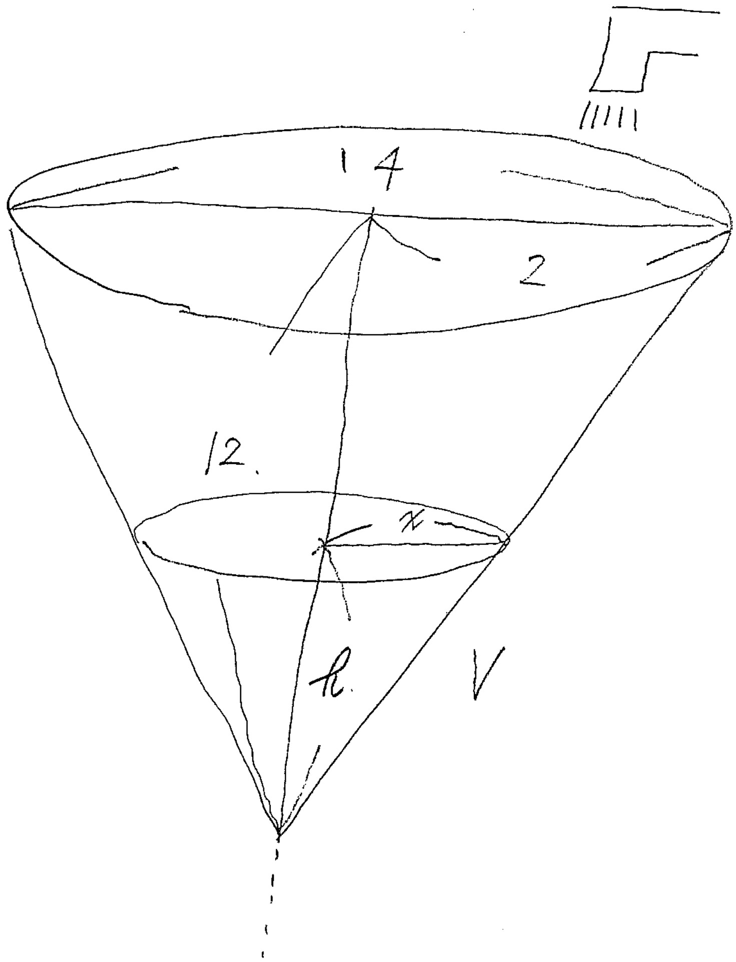
$$1 = -8 \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{8}$$

The angle is decreasing at a rate of $\frac{1}{8}$ rad/sec.

$\boxed{\text{Answer C} \quad \frac{1}{8} \text{ rad/sec}}$

11.



Given.

$$\frac{dV}{dt} = r - 10000$$

$$\frac{dh}{dt} = 20 \quad \text{when} \quad h = 200$$

Unknown.

$$r = ?$$

Relation.

$$V = \frac{1}{3}\pi x^2 \cdot h = \frac{1}{3}\pi\left(\frac{h}{6}\right)^2 \cdot h$$

$$= \frac{\pi}{108} h^3$$

Solution

$$\frac{dV}{dt} = \frac{\pi}{108} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{108} \cdot 3 \cdot (200)^2 \cdot 20$$

$$= \frac{200000}{9} \pi$$

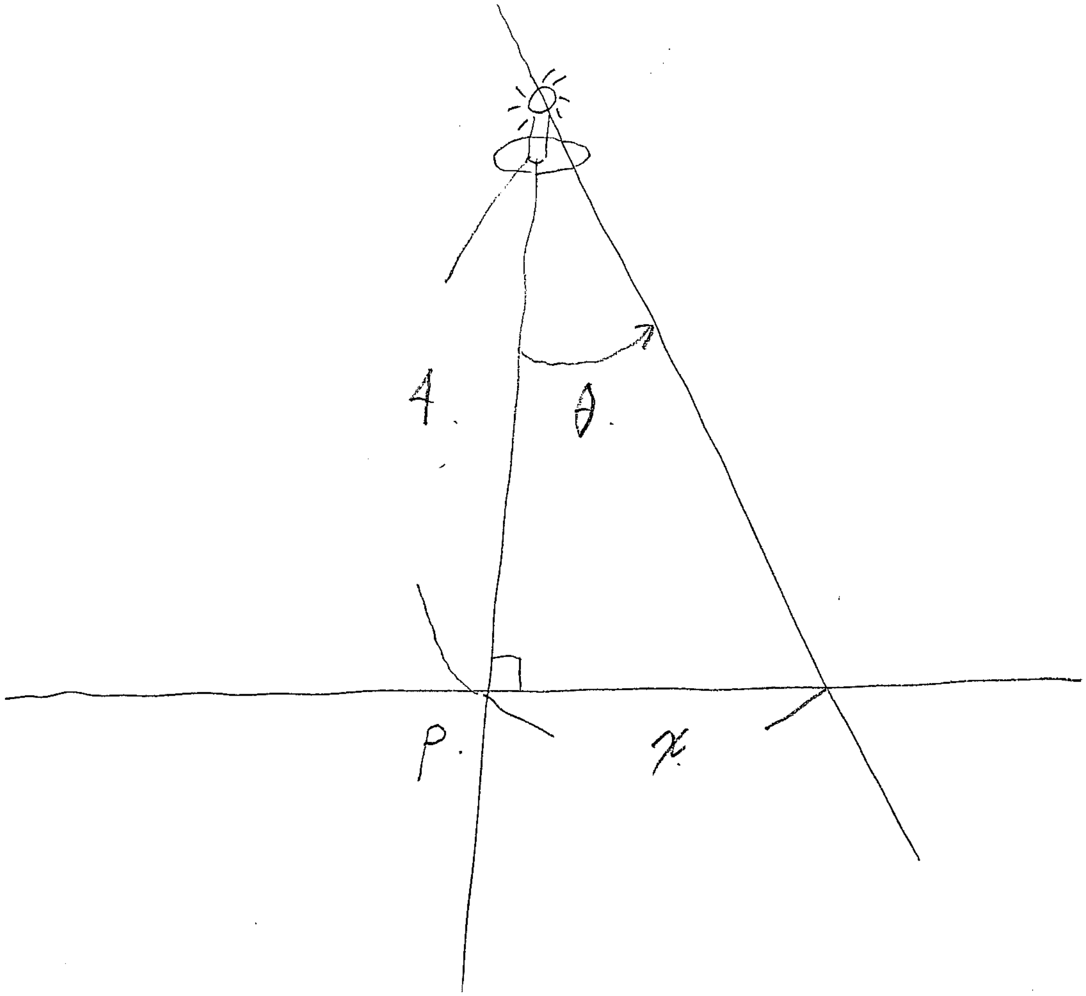
$$\therefore \quad V - 10000 = \frac{200000}{9} \pi$$

$$V = 10000 + \frac{200000}{9} \pi$$

$$= 10000 \left[1 + \frac{20}{9} \pi \right]$$

Answer D. $10000 \left[1 + \frac{20\pi}{9} \right]$ cm^3/min.

12.



Given

$$\frac{d\theta}{dt} = 10\pi$$

Unknown

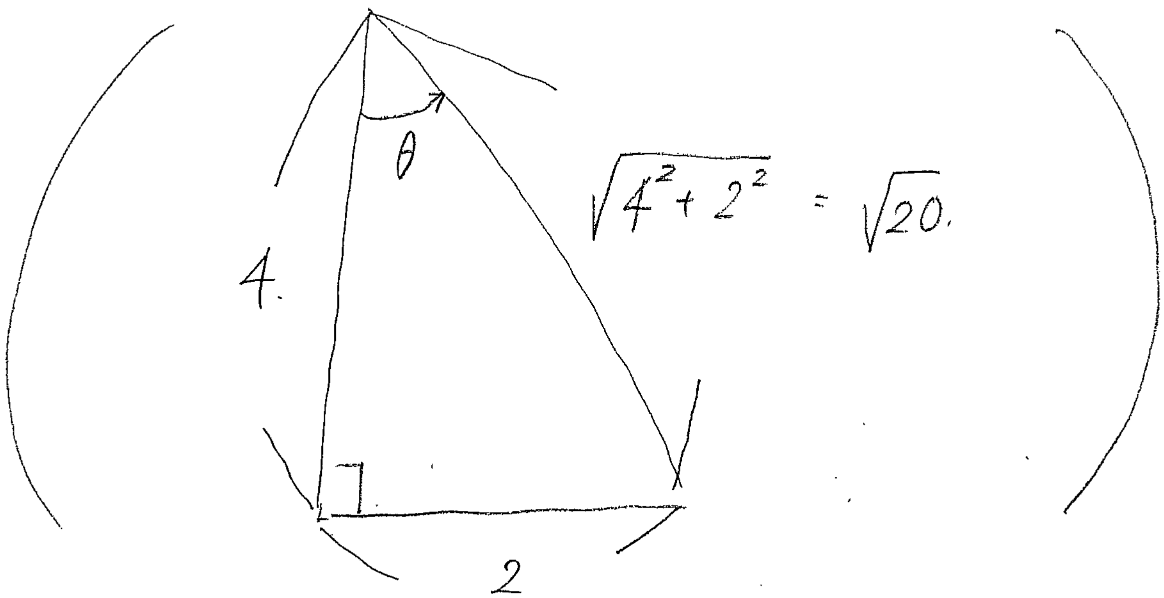
$$\frac{dx}{dt} = ? \quad \text{when} \quad x = 2$$

Relation

$$\frac{x}{4} = \tan\theta \quad \text{i.e.} \quad x = 4\tan\theta$$

Solution

$$\frac{dx}{dt} = 4 \sec^2\theta \cdot \frac{d\theta}{dt}$$



$$= 4 \cdot \left(\frac{\sqrt{20}}{4}\right)^2 \cdot 10 \cdot \pi$$

$$= 50\pi$$

$$\boxed{\text{Answer A. } 50\pi \text{ km/min.}}$$

Answer Keys for Exam 2 Version 02.

1. $F(\theta) = \sin^{-1}\left(\sqrt{\sin\theta}\right)$

Set

$$\begin{cases} u = \sin\theta \\ v = \sqrt{u}. \\ w = \sin^{-1} v. \end{cases}$$

Then

$$F'(\theta) = \frac{dF}{d\theta} = \frac{dw}{d\theta}$$

$$= \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{d\theta}$$

$$= \frac{1}{\sqrt{1-v^2}} \frac{1}{2\sqrt{u}} \cdot \cos\theta$$

$$= \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta$$

$$= \frac{\cos\theta}{2\sqrt{1-\sin\theta} \cdot \sqrt{\sin\theta}}$$

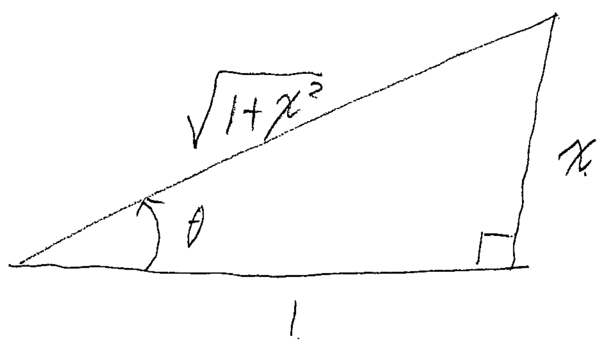
$$\boxed{\text{Answer } B, \quad \frac{\cos\theta}{2\sqrt{1-\sin\theta}\sqrt{\sin\theta}}}$$

2. $\sin(\tan^{-1} x)$

Set $\tan^{-1} x = \theta$

Then $x = \tan\theta$

Consider the following triangle



Then we have

$$\sin(\tan^{-1} x) = \sin\theta$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$\boxed{\text{Answer E. } \frac{x}{\sqrt{1+x^2}}}$$

3. $f(x) + x^3 [f(x)]^2 = \cancel{5}\ 6$

Then

$$f'(x) + 3x^2 [f(x)]^2 + x^3 \cdot 2 f(x) \cdot f'(x) = 0$$

Therefore

$$f'(1) + 3 \cdot 1^2 \cdot [f(1)]^2 + 1^3 \cdot 2 f(1) \cdot f'(1) = 0$$

Since $f(1) = 2$, we have

$$f'(1) + 3 \cdot 1^2 \cdot 2^2 + 1^3 \cdot 2 \cdot 2 \cdot f'(1) = 0$$

i.e.

$$5 f'(1) + 12 = 0$$

$$\therefore \quad f'(1) = -\frac{12}{5}.$$

$\boxed{\text{Answer B. } -\frac{12}{5}}$

4.

$$y \sin(2x) = x \cos(2y)$$

Apply $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{dy}{dx} \sin(2x) + y \cos(2x) \cdot 2$$
$$= 1 \cdot \cos(2y) + x \{-\sin(2y)\} \cdot 2 \frac{dy}{dx}$$

Plug in $(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ to the above equation to obtain

$$\cancel{\frac{dy}{dx} \sin(\pi)} + \frac{\pi}{4} \cos(\pi) \cdot 2$$
$$= 1 \cdot \cancel{\cos\left(\frac{\pi}{2}\right)} + \frac{\pi}{2} \{-\sin\left(\frac{\pi}{2}\right)\} 2 \cdot \frac{dy}{dx}$$

$$-\frac{\pi}{2} = -\pi \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

$\boxed{\text{Answer C. } \frac{1}{2}.}$

5.
$$y = x^{\ln x}$$

$$\ln y = \ln\left(x^{\ln x}\right)$$

$$= \ln x \cdot \ln x$$

$$= (\ln x)^2$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \ln x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \cdot 2 \ln x \cdot \frac{1}{x}$$

$$= x^{\ln x} \cdot 2 \ln x \cdot \frac{1}{x}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=e} = e^{\ln e} \cdot 2 \ln e \cdot \frac{1}{e}$$

$$= e^1 \cdot 2 \cdot 1 \cdot \frac{1}{e} = 2$$

$\boxed{\text{Answer } B.\ 2.}$

6. The linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 8$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[3]{8} + \frac{1}{3\cdot(\sqrt[3]{8})^2}(x-8)$$

$$= 2 + \frac{1}{12}(x-8)$$

Therefore,

$$\sqrt[3]{8.1} = f(8.1)$$

$$\approx L(8.1)$$

$$= 2 + \frac{1}{12}(8.1-8)$$

$$= \frac{241}{120}$$

$$\boxed{\text{Answer D. } \frac{241}{120}}$$

7.

$$\{\ln(5\cosh x)\}'$$

$$= \frac{(5\cosh x)'}{5\cosh x}$$

$$= \frac{\cancel{5}\sinh x}{\cancel{5}\cosh x} = \tanh x$$

$$\boxed{\text{Answer } B.\ \tanh x}$$

8. $y = x^3 - 2x^2 + 1$

$$dy = \frac{dy}{dx} dx$$

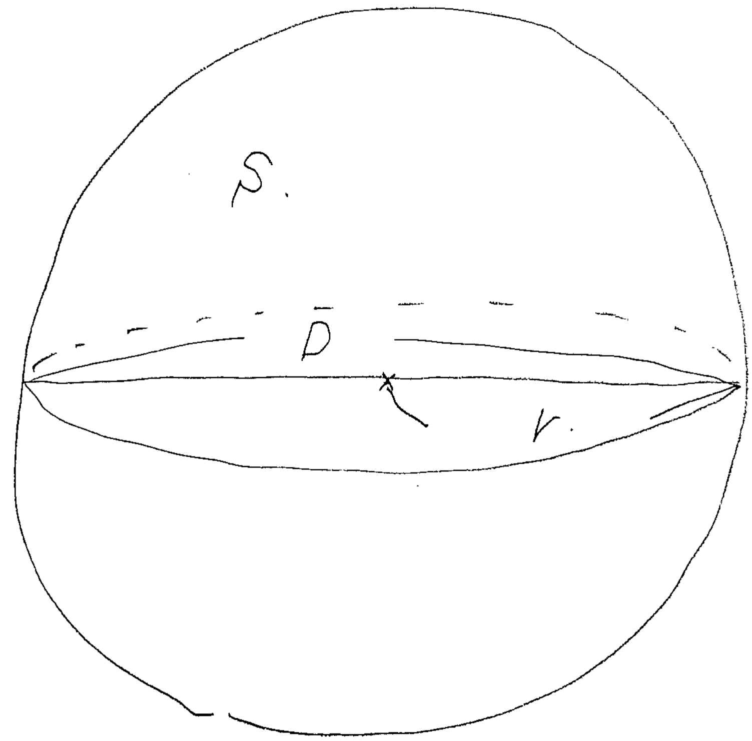
$$= (3x^2 - 4x)\, dx$$

$$= (3 \cdot 3^2 - 4 \cdot 3) \cdot 0.1$$

$$= 15 \cdot 0.1 = 1.5$$

$\boxed{\text{Answer C. } 1.5}$

9.



Given

$$\frac{dS}{dt} = -2$$

Unknown

$$\frac{dD}{dt} = ?$$

when $D = 10$

Relation

$$S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2$$

$$= \pi D^2$$

Solution

$$\frac{dS}{dt} = \pi \cdot 2D \cdot \frac{dD}{dt}$$

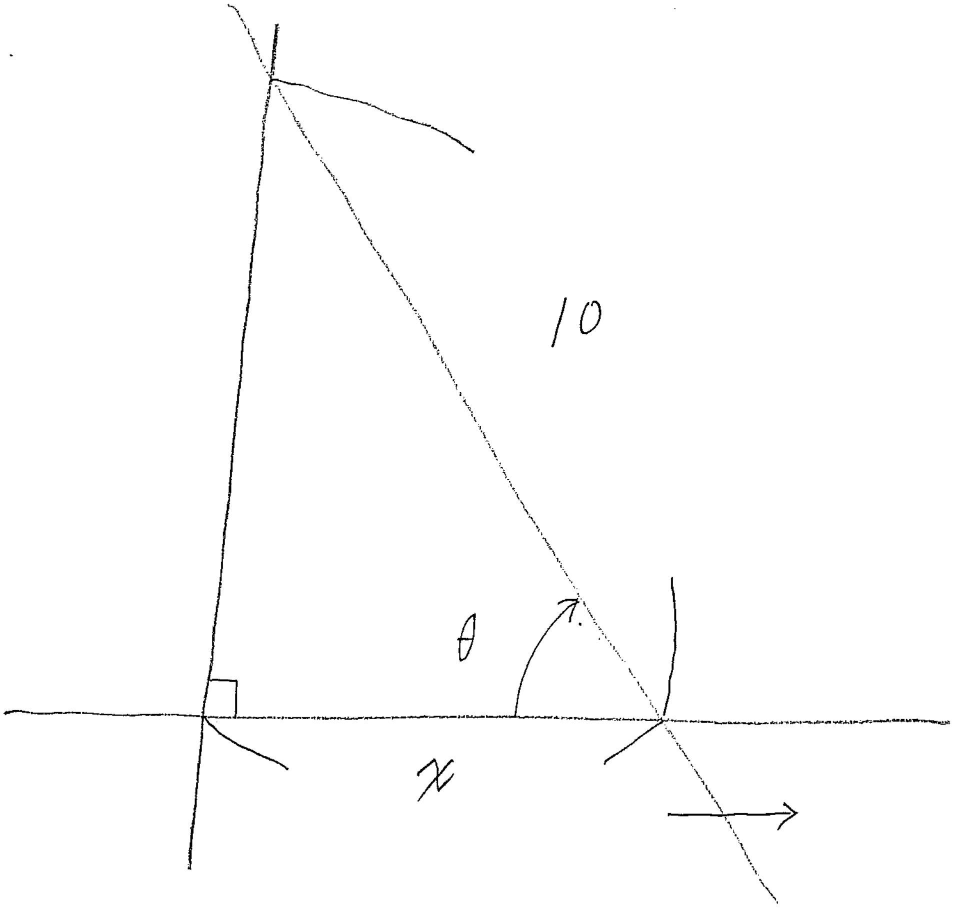
$$-2 = \pi \cdot 2 \cdot 10 \cdot \frac{dD}{dt}$$

$$\therefore \frac{dD}{dt} = -\frac{1}{10\pi}$$

The diameter is decreasing at a rate of $\frac{1}{10\pi}$ cm/min.

$$\boxed{\text{Answer B. } \frac{1}{10\pi} \text{ cm/min}}$$

10.



Given

$$\frac{dx}{dt} = 1.$$

Unknown.

$$\frac{d\theta}{dt} = ? \quad \text{when} \quad x = 8$$

Relation

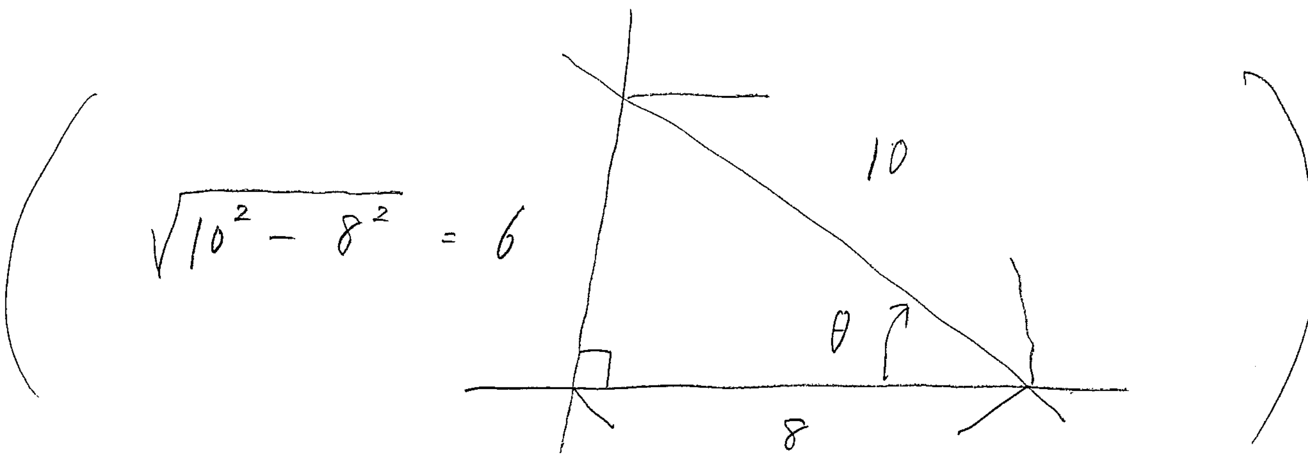
$$\frac{x}{10} = \cos\theta$$

i.e.

$$x = 10\cos\theta$$

Solution

$$\frac{dx}{dt} = -10 \sin\theta \cdot \frac{d\theta}{dt}$$



$$= -10 \cdot \frac{6}{10} \cdot \frac{d\theta}{dt}$$

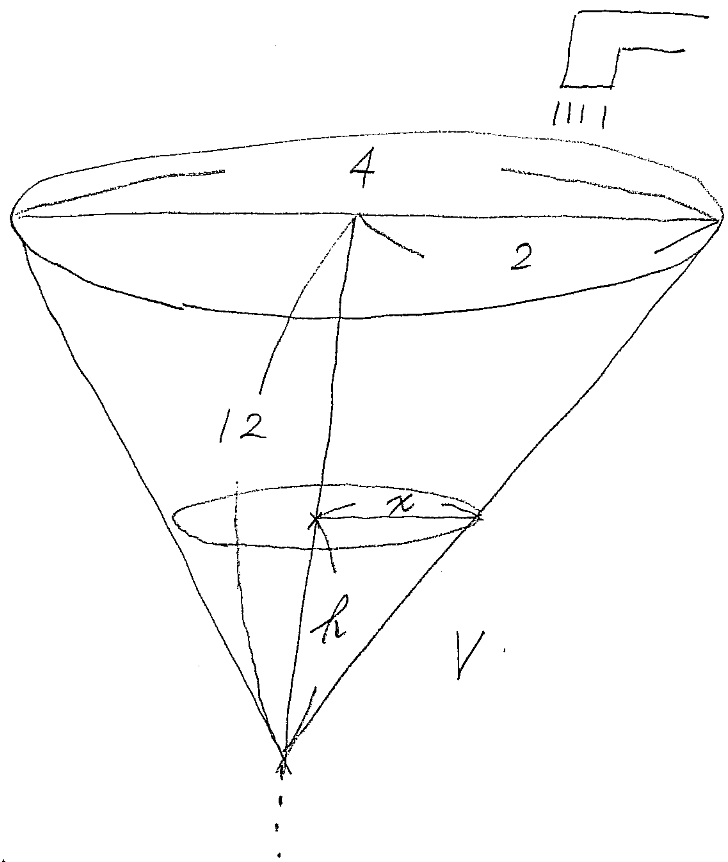
$$1 = -6 \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{6}$$

The angle is decreasing at a rate of $\frac{1}{6}$ rad/sec.

$$\boxed{\text{Answer E. } \frac{1}{6} \text{ rad/sec}}$$

11.



Given

$$\frac{dV}{dt} = r - 10000$$

$$\frac{dh}{dt} = 20 \quad \text{when } h = 200$$

Unknown

$$r = ?$$

Relation

$$V = \frac{1}{3}\pi x^2 \cdot h = \frac{1}{3}\pi\left(\frac{h}{6}\right)^2 h.$$

$$= \frac{\pi}{108} h^3$$

Solution

$$\frac{dV}{dt} = \frac{\pi}{108} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{108} \cdot 3 \cdot (200)^2 \cdot 20$$

$$= \frac{200000}{9} \pi .$$

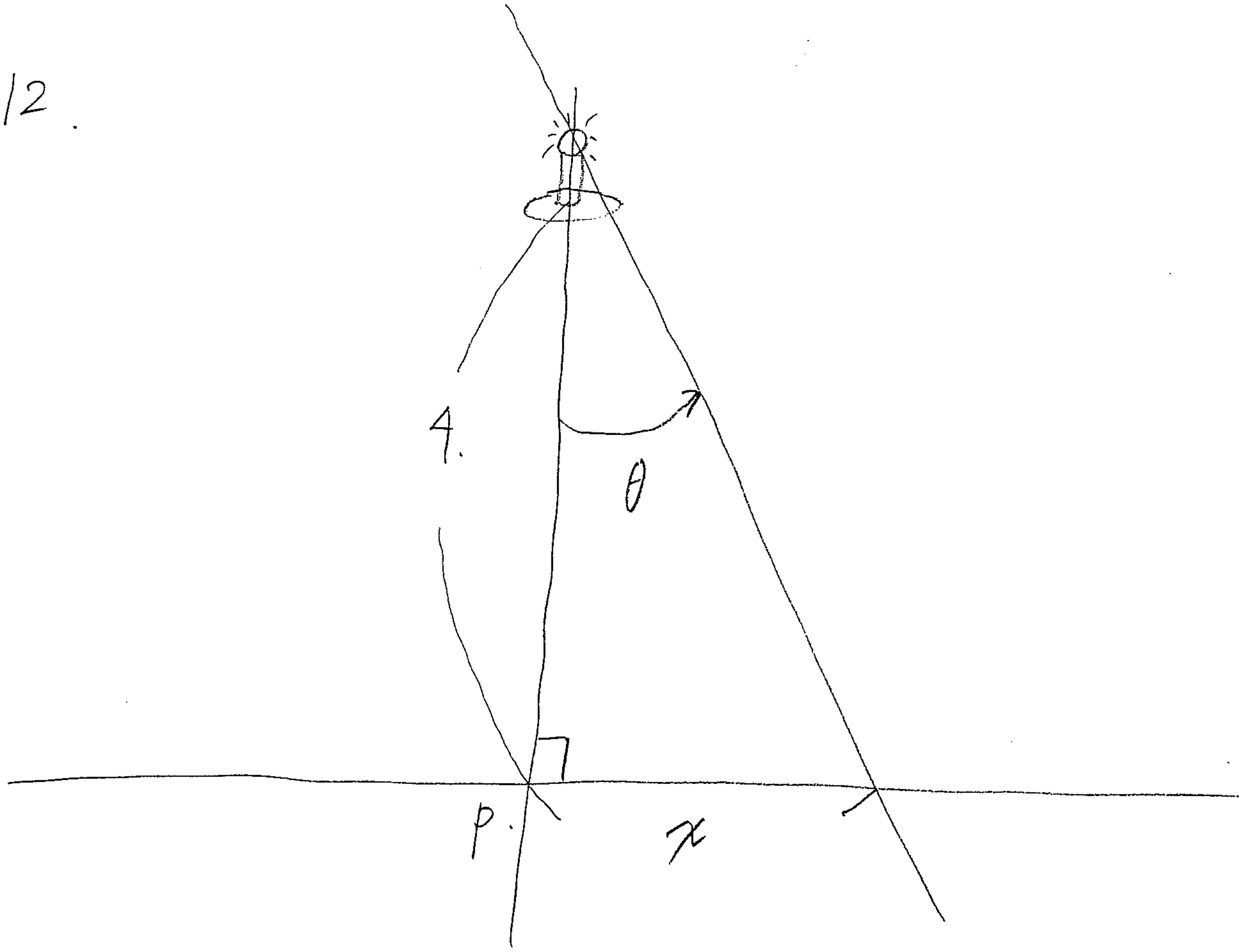
$$\therefore \quad r - 10000 = \frac{200000}{9} \pi$$

$$r = 10000 + \frac{200000}{9} \pi .$$

$$= 10000 \left[1 + \frac{20}{9} \pi \right]$$

Answer A. $10000 \left[1 + \frac{20}{9} \pi \right]$ $cm^3/min.$

12.



Given

$$\frac{d\theta}{dt} = 12\pi$$

Unknown

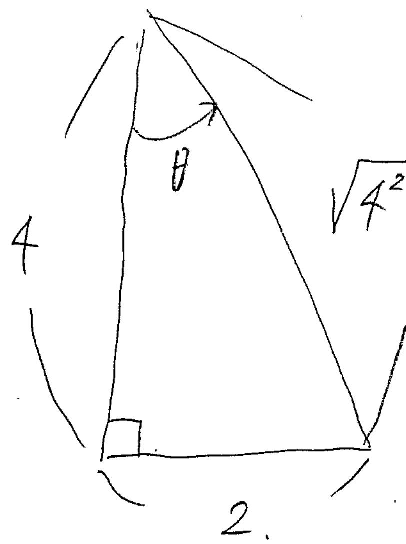
$$\frac{dx}{dt} = ? \quad \text{when} \quad x = 2.$$

Relation

$$\frac{x}{4} = \tan\theta \quad \text{i.e.} \quad x = 4\tan\theta$$

Solution

$$\frac{dx}{dt} = 4 \sec^2\theta \cdot \frac{d\theta}{dt}$$



$$\left(\sqrt{4^2+2^2} = \sqrt{20} \right)$$

$$= 4 \cdot \left(\frac{\sqrt{20}}{4}\right)^2 \cdot 12\pi$$

$$= 60\pi.$$

$\boxed{\text{Answer B. } 60\pi \text{ km/min.}}$