NAME GRADING KEY

10–digit PUID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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Page 2	/29
Page 3	/31
Page 4	/24
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3 and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators or any electronic devices may be used on this exam.

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(16) 1. Find the derivatives of the following functions. (It is not necessary to simplify).

(a) $y = e^{x\cos x}$

NPC.

$\begin{cases} u = x\cos x \\ y = e^u \end{cases}$ $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^u(\cos x - x\sin x) = e^{x\cos x}(\cos x - x\sin x)$

$\boxed{e^{x\cos x}(\cos x - x\sin x)}$ 4

(b) $y = \tan^{-1}\sqrt{x}$

$\begin{cases} u = \sqrt{x} \\ y = \tan^{-1} u \end{cases}$ $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{1+u^2}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$

$\boxed{\dfrac{1}{2(1+x)\sqrt{x}}}$ 4

(c) $f(x) = \sqrt[3]{1+\tan x}$.

$\begin{cases} u = 1+\tan x \\ y = \sqrt[3]{u} \end{cases}$ $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{3(\sqrt[3]{u})^2}\sec^2 x = \dfrac{\sec^2 x}{3(\sqrt[3]{1+\tan x})^2}$

$\boxed{\dfrac{\sec^2 x}{3(\sqrt[3]{1+\tan x})^2}}$ 4

or $\dfrac{1}{3}(1+\tan x)^{-\frac{2}{3}}\cdot\sec^2 x$

(d) $f(x) = \ln|2x+1|$

$\begin{cases} u = 2x+1 \\ y = \ln|u|. \end{cases}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{u}\cdot 2 = \dfrac{2}{2x+1}$

$\boxed{\dfrac{2}{2x+1}}$ 4

(8) 2. Find an equation of the tangent line to the curve $y = e^{\sin(\frac{\pi}{2}x)}$ at the point (2, 1).

$$\frac{dy}{dx} = e^{\sin(\frac{\pi}{2}x)} \cdot \cos\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2} \quad (3)$$

$$\left.\frac{dy}{dx}\right|_{x=2} = -\frac{\pi}{2} \quad (3)$$

Eq. of tan

$y - 1 = -\frac{\pi}{2}(x-2)$

$\boxed{y - 1 = -\frac{\pi}{2}(x-2)}$ (2)

concis

(7) 3. Use implicit differentiation to find the slope of the tangent line to the curve

$$x^2 + 2xy - y^2 + x = 2$$

at the point (1,2).

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 1 = 0 \quad (4)$$

$$(2x - 2y)\frac{dy}{dx} = -2x - 2y - 1$$

$$\frac{dy}{dx} = \frac{-2x-2y-1}{2x-2y} \qquad \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{7}{2}$$

$\boxed{\frac{7}{2}}$ (3)

(6) 4. Find the value for $f^{(4)}(1)$ (the 4–th derivative evaluated at $x = 1$) when $f(x) = \sqrt{2x-1}$.

(☆)

$$f(x) = \sqrt{2x-1} = (2x-1)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \cdot 2 = (2x-1)^{-\frac{1}{2}} \quad (1)$$

$$f''(x) = -\frac{1}{2}(2x-1)^{-\frac{3}{2}} \cdot 2 = -(2x-1)^{-\frac{3}{2}} \quad (1)$$

$$f'''(x) = -\left(-\frac{3}{2}\right)(2x-1)^{-\frac{5}{2}} \cdot 2 = 3(2x-1)^{-\frac{5}{2}} \quad (1)$$

$$f^{(4)}(x) = 3\left(-\frac{5}{2}\right)(2x-1)^{-\frac{7}{2}} \cdot 2 = -15(2x-1)^{-\frac{7}{2}} \quad (1)$$

$$f^{(4)}(1) = -15$$

(6) if the ans is correct. If not, look (☆), (1) for each step.

$\boxed{f^{(4)}(1) = -15}$ (6)

(8) 5. Find the derivative of the function $y = x^{\ln x}$.

$$y = x^{\ln x} = (e^{\ln x})^{\ln x} = e^{(\ln x)^2} \quad (4)$$

$$\frac{dy}{dx} = \boxed{e^{(\ln x)^2} \cdot 2\ln x \cdot \frac{1}{x}} \text{ or}$$

$$= x^{\ln x} \cdot 2\ln x \cdot \frac{1}{x}$$

$\boxed{\frac{dy}{dx} = x^{\ln x}\, 2\ln x \cdot \frac{1}{x}}$ (4)

or $\ln y = \ln x \cdot \ln x = (\ln x)^2$ (4) $\frac{1}{y}\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x}$

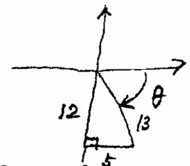
NPC

(9) 6. Find the exact value of each expression.

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad \boxed{\frac{\pi}{3}}$ 3

(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \theta$ $\quad \tan\theta = \tan\frac{4\pi}{3}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad \boxed{\frac{\pi}{3}}$ 3

(c) $\cos\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$ $\quad \sin^{-1}\left(-\frac{12}{13}\right) = \theta$



$\boxed{\frac{5}{13}}$ 3

(6) 7. Find the exact value of the derivative of $f(x) = \cosh x$ when $x = \ln 2$. Write your answer in the form of a ratio of two integers.

$$f(x) = \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$f'(x) = \sinh x = \frac{e^{x} - e^{-x}}{2}$$ 3

$$f'(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$$

$\boxed{\left.\frac{dy}{dx}\right|_{x=\ln 2} = \frac{3}{4}}$ 3

(10) 8. (a) Find the linearization $L(x)$ of the function $f(x) = \ln x$ at $a = 1$.

$$L(x) = f(a) + f'(a)(x-a)$$
$$= f(1) + f'(1)(x-1)$$ 3

$f(1) = 0 \quad f'(1) = 1$

$\boxed{L(x) = x - 1}$ 3

(b) Use a linear approximation to estimate the number $\ln(1.1)$.

$$\ln(1.1) \approx L(1.1)$$
$$= 1.1 - 1$$
$$= 0.1$$

$\boxed{\ln(1.1) \approx 0.1}$ 4

(6) 9. Let f be a function such that

$$f(1) = 3 \text{ and } f'(1) = 5$$

(a) Set $h(x) = \dfrac{1}{f(x)}$. Find $h'(1)$.

$$h'(x) = -\frac{f'(x)}{f(x)^2}$$

$$h'(1) = -\frac{f'(1)}{f(1)^2} = -\frac{5}{3^2} = -\frac{5}{9}$$

$\boxed{-\frac{5}{9}}$ 3

(b) Let g be the inverse function of f (so that $(g \circ f)(x) = x$). Find $g'(3)$.

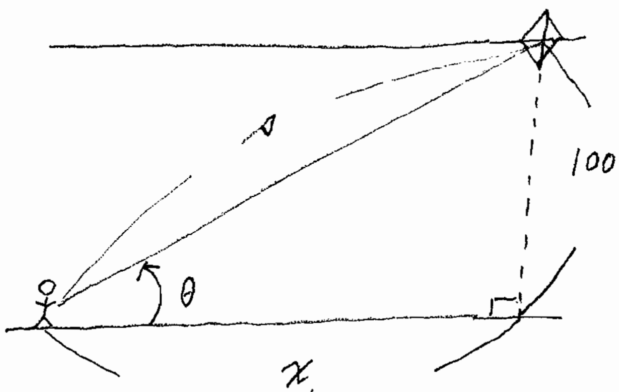
$$(g \circ f)'(x) = g'(f(x))\, f'(x) = (x)' = 1$$

$$\therefore \quad g'(f(1))\, f'(1) = 1. \rightarrow g'(3) \cdot 5 = 1$$
$$g'(3) = \frac{1}{5}$$

$\boxed{\frac{1}{5}}$ 3

Also see alternate solutions on next page

(12) 10. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?



Given $\frac{dx}{dt} = 8$ ①

Unknown $\frac{d\theta}{dt}$ when $s = 200$

Relation

$x \tan\theta = 100$ ④

Solution (Differentiation)

$$\frac{dx}{dt} \cdot \tan\theta + x \sec^2\theta \frac{d\theta}{dt} = 0 \quad ④$$

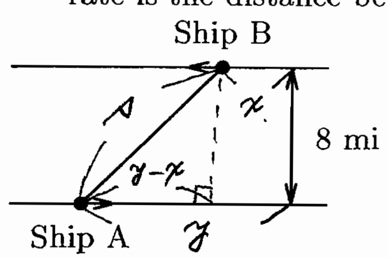
When $s = 200$:

$$\begin{cases} \tan\theta = \frac{1}{\sqrt{3}} \quad \sec\theta = \frac{2}{\sqrt{3}} \\ x = 100\sqrt{3} \end{cases}$$

$$\frac{d\theta}{dt} = -\frac{8 \cdot \frac{1}{\sqrt{3}}}{100\sqrt{3}\left(\frac{2}{\sqrt{3}}\right)^2} = -\frac{1}{50}$$

$\boxed{\text{The angle decreases at } \frac{1}{50} \text{ rad/s.}}$ ③

(12) 11. Two ships, one heading west and the other east, approach each other on parallel courses 8 miles apart. Given that each ship is cruising at 20 miles per hour, at what rate is the distance between them diminishing when they are 10 miles apart?



Given $\frac{dx}{dt} = 20$ $\frac{dy}{dt} = -20$. ③

Unknown $\frac{ds}{dt}$ when $s = 10$.

Relation:

$$s^2 = 8^2 + (y-x)^2 \quad ③$$

Solution (Differentiation)

$$2s\frac{ds}{dt} = 2(y-x)\left(\frac{dy}{dt} - \frac{dx}{dt}\right) \quad ③$$

When $s=10$:

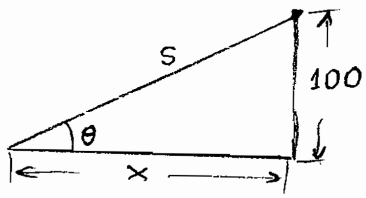
$$\begin{cases} 10^2 = 8^2 + (y-x)^2 \\ \therefore\ y - x = 6 \end{cases}$$

$$2 \cdot 10 \frac{ds}{dt} = 2 \cdot 6(-20-20)$$

$$\therefore \frac{ds}{dt} = -24.$$

$\boxed{\text{The distance is diminishing at 24 miles/h.}}$ ③

(12) 10. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?



Given $\frac{dx}{dt} = 8$ ft/sec (1)

Find $\frac{d\theta}{dt}$ when $s = 200$ ft

$x \tan\theta = 100$ (4)

$\frac{dx}{dt}\tan\theta + x\sec^2\theta \frac{d\theta}{dt} = 0$ (4)

When $s = 200$:

$x = \sqrt{200^2 - 100^2} = 100\sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}}$, $\sec\theta = \frac{2}{\sqrt{3}}$

$8 \cdot \frac{1}{\sqrt{3}} + 100\sqrt{3}\left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = 0$

$\frac{d\theta}{dt} = -\frac{8\frac{1}{\sqrt{3}}}{100\sqrt{3}\left(\frac{2}{\sqrt{3}}\right)^2} = -\frac{1}{50}$

or

$\cot\theta = \frac{x}{100}$ (4)

$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt}$ (4)

When $s = 200$

$\csc\theta = \frac{1}{\sin\theta} = \frac{200}{100} = 2$

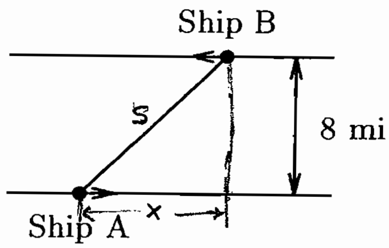
$-4\frac{d\theta}{dt} = \frac{1}{100}8$

$\frac{d\theta}{dt} = -\frac{1}{50}$

(3)

$\boxed{\text{The angle decreases at } \frac{1}{50} \text{ rads/sec}}$

(12) 11. Two ships, one heading west and the other east, approach each other on parallel courses 8 miles apart. Given that each ship is cruising at 20 miles per hour, at what rate is the distance between them diminishing when they are 10 miles apart?



Since the two ships are approaching each other on parallel courses and each is cruising at 20 mi/hr,

$$\frac{dx}{dt} = -40 \text{ mi/hr} \quad (3)$$

Find $\frac{ds}{dt}$ when $s = 10$ mi

$$x^2 + 8^2 = s^2 \quad (3)$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt} \quad (3)$$

When $s = 10$

$x^2 = 10^2 - 8^2$, $x = 6$

$2 \cdot 6 \cdot (-40) = 2 \cdot 10 \cdot \frac{ds}{dt}$

$\frac{ds}{dt} = -24$ mi/hr

(3)

$\boxed{\text{The distance is diminishing at 24 mi/hr}}$