MATH 162 – SPRING 2010 – FINAL EXAM – MAY 7, 2010

VERSION 01

MARK TEST NUMBER 01 ON YOUR SCANTRON

STUDENT NAME SOLUTIONS

STUDENT ID

RECITATION INSTRUCTOR

INSTRUCTOR

RECITATION TIME

INSTRUCTIONS

1. Fill in all the information requested above and the version number of the test on your scantron sheet.
2. This booklet contains 25 problems, each one is worth 8 points. The maximum score is 200 points.
3. For each problem mark your answer on the scantron sheet and also circle it in this booklet.
4. Work only on the pages of this booklet.
5. Books, notes and calculators are not allowed.
6. At the end turn in your exam and scantron sheet to your recitation instructor.

1) The area of the triangle with vertices $P(1,2,1)$, $Q(-1,3,2)$ and $R(3,1,1)$ is equal to

A) 2

B) $4\sqrt{2}$

C) $\frac{\sqrt{3}}{2}$

D) $\frac{\sqrt{5}}{2}$

E) $2\sqrt{3}$

$\vec{PQ} = \langle -2, 1, 1 \rangle$ and $\vec{PR} = \langle 2, -1, 0 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = \langle 1, -2, 0 \rangle$$

$$\text{area of triangle} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{1^2 + (-2)^2 + 0^2} = \frac{\sqrt{5}}{2}$$

2) Let $P(2,4)$, $Q(3,-1)$ and $R(1,3)$ be 3 points. The cosine of the angle between vectors $\vec{PQ}$ and $\vec{QR}$ is

A) $\frac{-3}{\sqrt{52}}$

B) $\frac{2}{\sqrt{40}}$

C) $\frac{-2}{\sqrt{40}}$

D) $\frac{3}{\sqrt{52}}$

E) $\frac{-22}{\sqrt{520}}$

$\vec{PQ} = \langle 1, -5 \rangle$ and $\vec{QR} = \langle -2, 4 \rangle$

$$\cos\theta = \frac{\vec{PQ} \cdot \vec{QR}}{|\vec{PQ}|\,|\vec{QR}|} = \frac{-2-20}{\sqrt{26}\sqrt{20}} = \frac{-22}{\sqrt{520}}$$

3) The area of the region bounded by the curves $y = 2 - x^2$ and $y = x$ is

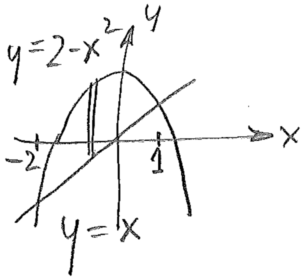
A) $\frac{42}{5}$

B) 6

C) $\frac{37}{4}$

D) $\frac{9}{2}$

E) $\frac{38}{3}$



intersection : $2 - x^2 = x$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$\rightarrow x = -2, 1$

$$\text{Area} = \int_{-2}^{1} \left[(2-x^2)-(x)\right] dx$$

$$= \left(2x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_{-2}^{1}$$

$$= \left(2 - \tfrac{1}{3} - \tfrac{1}{2}\right) - \left(-4 + \tfrac{8}{3} - 2\right) = \tfrac{9}{2}$$

4) The region bounded by $y = 2x$, $y = 0$ and $x = 2$ is rotated about the y–axis. The volume of the resulting solid of revolution (using the disk/washer method) is

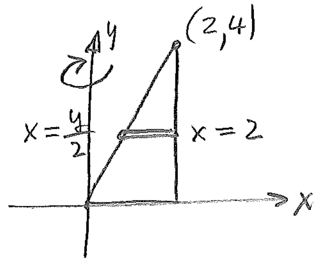
A) $\displaystyle\int_0^4 \pi\left(4 - \left(\frac{y}{2}\right)^2\right) dy$

B) $\displaystyle\int_0^4 \pi\left(2 - \frac{y}{2}\right)^2 dy$

C) $\displaystyle\int_0^4 \pi(2x)^2\, dx$

D) $\displaystyle\int_0^2 2\pi(2x)\, dx$

E) $\displaystyle\int_0^2 2\pi((2x)^2 - 2)\, dx$



$$\text{Volume} = \int_0^4 \pi\left[2^2 - \left(\tfrac{y}{2}\right)^2\right] dy$$

5) The region of the first quadrant bounded by the curves $y = x$ and $y = \sqrt{x}$ is rotated about the axis $x = 1$. The volume of the resulting solid of revolution (using the cylindrical shells method) is equal to

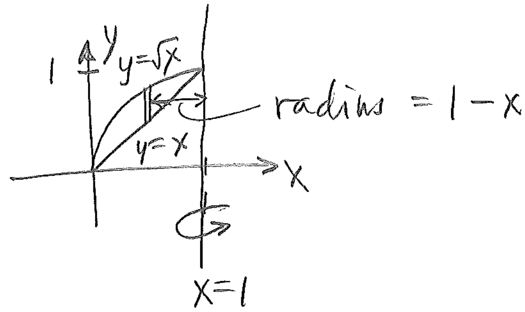
A) $2\pi \int_0^1 x(\sqrt{x} - x)\, dx$

B) $2\pi \int_0^1 (1 - x)(\sqrt{x} - x)\, dx$

C) $2\pi \int_0^1 (1 - 2x)(\sqrt{x} - x)\, dx$

D) $2\pi \int_0^1 (1 - x)(x - \sqrt{x})\, dx$

E) $2\pi \int_0^1 x(x - \sqrt{x})\, dx$



$$\text{Volume} = \int_0^1 2\pi(1-x)(\sqrt{x}-x)\,dx$$

6) If the work required to stretch a spring 1/2 ft beyond its natural length is 8 ft–lbs, how much work is needed to stretch it 1/3 ft beyond its natural length?

A) $\frac{4}{9}$ ft–lbs

B) $\frac{32}{9}$ ft-lbs

C) 24 ft–lbs

D) $\frac{8}{3}$ ft–lbs

E) $\frac{8}{6}$ ft–lbs

$$\int_0^{1/2} kx\,dx = 8 \text{ ft-lbs}$$

$$\rightarrow \frac{k}{2}x^2\Big|_0^{1/2} = 8 \rightarrow \frac{k}{8} = 8 \rightarrow k = 64$$

$$\rightarrow F(x) = 64x$$

$$\int_0^{1/3} 64x\,dx = 32x^2\Big|_0^{1/3} = \frac{32}{9} \text{ ft-lbs}$$

$$\int u\,dv = uv - \int v\,du$$

let u = function first in $L^{og}\ I^{nv\,trig}\ A^{lg}\ T^{rig}\ E^{xp}$

7) $\displaystyle\int_2^{\ln 10} xe^x\,dx =$

A) $\ln 10^9 - e^2$

B) $90 + e^2$

C) $90 - e^2$

D) $\ln 10^{10} + 3e^2$

E) $\ln 10^{10} - 10 - e^2$

let $u = x$ and $dv = e^x\,dx$

then $du = dx$ and $v = e^x$

$$\int_2^{\ln 10} xe^x\,dx = xe^x\Big|_2^{\ln 10} - \int_2^{\ln 10} e^x\,dx$$

$$= (xe^x - e^x)\Big|_2^{\ln 10}$$

$$= \big((\ln 10)(10) - 10\big) - (2e^2 - e^2)$$

$$= 10\ln 10 - 10 - e^2$$

$$= \ln 10^{10} - 10 - e^2$$

8) $\displaystyle\int_0^{\pi/6} \sin x\cos^3 x\,dx = *$

A) $\frac{1}{64}$

B) $\frac{1}{4}$

C) $\frac{7}{64}$

D) $\frac{-9}{64}$

E) $\frac{-7}{64}$

let $u = \cos x$. Then $du = -\sin x\,dx$

and $-du = \sin x\,dx$

$u(0) = \cos 0 = 1$, $u(\pi/6) = \cos\pi/6 = \sqrt{3}/2$

$$* = \int_1^{\frac{\sqrt{3}}{2}} u^3(-du) = -\frac{1}{4}u^4\Big|_1^{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{4}\left(\frac{9}{16} - 1\right)$$

$$= -\frac{9}{64} + \frac{1}{4}$$

$$= -\frac{9}{64} + \frac{16}{64} = \frac{7}{64}$$

9) Which integral arises when one uses a trigonometric substitution to evaluate

$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx \quad = *$$

A) $\int 4\sin^2\theta\, d\theta$

B) $\int 4\sec^3\theta\, d\theta$

C) $\int 4\tan^2\theta\sec\theta\, d\theta$

D) $\int 4\tan\theta\sec^2\theta\, d\theta$

E) $\int 4\sec^2\theta\, d\theta$

Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\, d\theta$

and $\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta$

$$* = \int \frac{(2\sec\theta)^2}{2\tan\theta}\cdot 2\sec\theta\tan\theta\, d\theta$$

$$= \int 4\sec^3\theta\, d\theta$$

10) $\int \frac{2x-1}{x^2(x-2)}\,dx =$

A) $\frac{-5}{4}\ln|x| - \frac{3}{4}\ln|x-2| + \frac{1}{x} + C$

B) $\frac{5}{4}\ln|x| + \frac{3}{4}\ln|x-2| + \frac{1}{x} + C$

C) $\frac{-3}{4}\ln|x| + \frac{3}{4}\ln|x-2| - \frac{1}{2x} + C$

D) $\frac{3}{4}\ln|x| - \frac{5}{4}\ln|x-2| - \frac{1}{x} + C$

E) $\frac{-5}{4}\ln|x| + \frac{3}{4}\ln|x-2| - \frac{1}{x} + C$

$$\frac{2x-1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$\to 2x-1 = A(x)(x-2) + B(x-2) + C(x^2)$

$x=0 \to -1 = 0A - 2B + 0C \to B = \frac{1}{2}$

$x=2 \to 3 = 0A + 0B + 4C \to C = \frac{3}{4}$

$x=1,\ B=\frac{1}{2},\ C=\frac{3}{4} \to 1 = -A - \frac{1}{2} + \frac{3}{4} \to A = -\frac{3}{4}$

$$\int \left(\frac{-\frac{3}{4}}{x} + \frac{\frac{1}{2}}{x^2} + \frac{\frac{3}{4}}{x-2} \right) dx$$

$$= -\frac{3}{4}\ln|x| - \frac{1}{2}\cdot\frac{1}{x} + \frac{3}{4}\ln|x-2| + C$$

11) $\int_1^{\infty} \frac{\pi}{x^2}\,dx =$ $\lim_{t \to \infty} \int_1^t \frac{\pi}{x^2}\,dx = \lim_{t \to \infty} \left(-\frac{\pi}{x} \Big|_1^t \right)$

$= \lim_{t \to \infty} \left(-\frac{\pi}{t} + \pi \right) = 0 + \pi = \pi$

A) the integral diverges

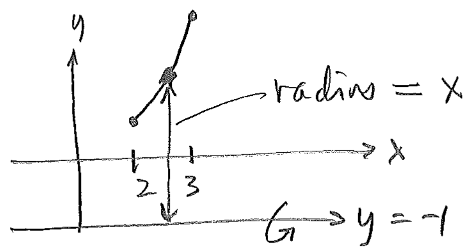
B) $\pi \ln 2$

C) $\pi \ln\left(\frac{1}{2}\right)$

D) π

E) 2π

12) The curve $y = x^2$, $2 \le x \le 3$ is rotated about the line $y = -1$. The resulting surface has area given by



$\frac{dy}{dx} = 2x$

A) $\int_2^3 2\pi(x^2 - 1)\sqrt{1 + x^4}\,dx$

B) $\int_2^3 2\pi(x + 1)\sqrt{1 + 4x^2}\,dx$

C) $\int_2^3 2\pi(x)\sqrt{1 + 4x^2}\,dx$

D) $\int_2^3 2\pi(x^2 + 1)\sqrt{1 + 4x^2}\,dx$

E) $\int_2^3 2\pi(x^2 - 1)\sqrt{1 + 4x^2}\,dx$

S.A. $= \int_2^3 2\pi\left(x^2+1\right)\sqrt{1+(2x)^2}\,dx$

13) The area of the region of the first quadrant bounded by $y = 2 - x^2$, $y = x$ and the y-axis is equal to $\frac{7}{6}$. Find the x-coordinate of the centroid of the region.

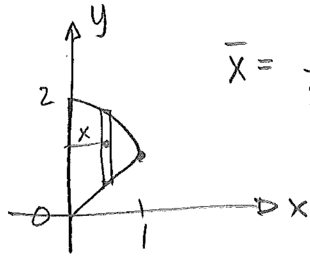
A) 7/12

B) 3/8

C) 5/8

D) 4/9

E) 5/14



$\bar{X} = \dfrac{M_y}{\frac{7}{6}}$

intersection : $2 - x^2 = x$

$0 = x^2 + x - 2$

$= (x+2)(x-1)$

$$M_y = \int_0^1 x(2 - x^2 - x)\,dx$$
$$= \int_0^1 (2x - x^3 - x^2)\,dx$$
$$= \left(x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= 1 - \tfrac{1}{4} - \tfrac{1}{3} = \frac{12-3-4}{12} = \frac{5}{12}$$

$$\bar{X} = \frac{M_y}{\frac{7}{6}} = \frac{6}{7}\left(\frac{5}{12}\right) = \frac{5}{14}$$

14) The limit of the sequence $a_n = n \sin\left(\frac{1}{n}\right)$ is equal to

A) 0

B) 1

C) 2

D) 3

E) 4

$$\lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

$\left(\frac{1}{n} \to 0\right)$

15) Which of the following statements are true about the series $\sum_{n=0}^{\infty} a_n$?

I) If $\lim_{n\to\infty} na_n = 1$, the series converges. False. $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = 1$ and $\sum^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum^{\infty} a_n$ also diverges

II) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the series converges.

III) If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, the series diverges.

$\Big\}$ False.

A) All three are correct

B) All three are incorrect

C) I and II are correct, III is false

D) II and III are correct, I is false

E) I and III are correct, II is false

The limit of 1 in the Ratio and Root Test is inconclusive.

Examples: $\sum^{\infty} \frac{1}{n}$ diverges

and $\sum^{\infty} \frac{1}{n^2}$ converges.

Both of these series have limit 1 in Ratio and Root Tests.

16) What can be said about the convergence of the following series

$$S_1 = \sum_{n=1}^{\infty} n \sin\left(\frac{1}{n^3}\right), \quad S_2 = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}, \quad S_3 = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}?$$

A) S_1 and S_2 converge, S_3 diverges

B) S_1 and S_3 diverge, S_2 converges

C) S_1, S_2 and S_3 converge

D) S_1, S_2 and S_3 diverge

E) S_1 and S_3 diverge, S_2 converges

S_1 converges:

$$\left|n \sin\left(\frac{1}{n^3}\right)\right| = \left(\frac{1}{n^2}\right)\left|\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n}\left(\frac{1}{n^2}\right)}\right| < \frac{1}{n^2}$$

and $\sum^{\infty} \frac{1}{n^2}$ converges.

$\therefore S_1$ converges absolutely.

S_2 converges: $\frac{\ln n}{n^2} < \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ converges $(p = \frac{3}{2} > 1)$

S_3 converges: Alt. Series Test. $\left(\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \text{ and } \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}\right)$

use Comparison Test

S_3 converges: $\frac{1}{n^2 \ln n} < \frac{1}{n^2}$ for $n > e$ and $\sum^{\infty} \frac{1}{n^2}$ conv.

Therefore $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges and so $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges

17) Which of the following series diverge?

$$S_1 = \sum_{n=1}^{\infty} \frac{n^2+1}{n^3}, \quad S_2 = \sum_{n=1}^{\infty} (-1)^n \frac{n^2+n}{n^3+n^2+n}, \quad S_3 = \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

A) S_1 only.

S_1 diverges: $\lim_{n\to\infty} \frac{\frac{n^2+1}{n^3}}{\frac{1}{n}} = 1 > 0$ and $\sum \frac{1}{n}$ diverges

$\rightarrow$ S_1 diverges by Limit Comparison Test.

B) S_2 only.

C) S_1 and S_2 only.

S_2 converges: $\lim_{n\to\infty} \frac{n^2+n}{n^3+n^2+n} = 0$.

D) S_2 and S_3 only.

Let $f(x) = \frac{x^2+x}{x^3+x^2+x}$. Note $f(x) = \frac{x+1}{x^2+x+1}$ for $x \neq 0$,

E) All of them.

$f'(x) = \frac{(1)(x^2+x+1) - (x+1)(2x+1)}{(x^2+x+1)^2} = \frac{-(x^2+2)}{(x^2+x+1)^2} < 0$,

Therefore S_2 converges by Alternating Series Test.

18) Which statement is true about the following series

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{3}}}, \quad S_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}, \quad S_3 = \sum_{n=1}^{\infty} (-1)^n \sin(\frac{n\pi}{2}) ?$$

A) All are conditionally convergent.

B) All are divergent.

C) S_1 is conditionally convergent, S_2 is absolutely convergent and S_3 is divergent

D) S_1 is absolutely convergent, S_2 is conditionally convergent and S_3 diverges

E) S_1 and S_2 are conditionally convergent; S_3 is absolutely convergent.

S_1 cond. conv. $\sum^{\infty} \frac{(-1)^n}{n^{1/3}}$ conv. (Alt Ser. Test); $\sum \frac{1}{n^{1/3}}$ div. (p-series $p = 1/3 < 1$)

S_2 abs. conv. $\sum^{\infty} \frac{1}{n^4}$ conv. (p-series. $p = 4 > 1$)

S_3 diverges. $\lim_{n\to\infty} (-1)^n \sin(\frac{n\pi}{2}) \neq 0$ Note: $\sin(\frac{n\pi}{2}) = \pm 1$, $n = 1, 2, 3, \ldots$

19) The radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{(n+1)^3}$ satisfy

A) The radius is equal to 1 and the interval is $(0,1)$.

B) The radius is equal to 2 and the interval is $(0,2)$.

C) The radius is equal to 1 and the interval is $(1,3)$.

D) The radius is equal to 1 and the interval is $(1,3]$.

E) The radius is equal to 1 and the interval is $[0,2]$.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+2)^3} \cdot \frac{(n+1)^3}{(x-1)^n} \right|$

$= \lim_{n\to\infty} |x-1| \frac{(n+1)^3}{(n+2)^3} = |x-1|$

and $|x-1| < 1 \rightarrow -1 < x-1 < 1$

$\rightarrow 0 < x < 2$

$\rightarrow$ radius of convergence is 1.

Convergence at endpts:

$x = 0 \Rightarrow \sum^{\infty} \frac{(-1)^n(-1)^n}{(n+1)^3} = \sum^{\infty} \frac{1}{(n+1)^3}$ conv. (Limit Comp. with conv. p-series $\sum^{\infty} \frac{1}{n^3}$)

$x = 2 \Rightarrow \sum^{\infty} \frac{(-1)^n(1)^n}{(n+1)^3}$ conv. by alt. Series Test.

20) Let $f(x) = \sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-1)^n$. We can say that the third derivative of f at the point 1 is equal to

A) $f^{(3)}(1) = 10$.

B) $f^{(3)}(1) = \frac{14}{5}$.

C) $f^{(3)}(1) = \frac{13}{6}$.

D) $f^{(3)}(1) = \frac{16}{3}$.

E) $f^{(3)}(1) = \frac{1}{9}$.

$\frac{f^{(n)}(1)}{n!} = \frac{2^n}{n^2}$

$\Rightarrow \frac{f^{(3)}(1)}{3!} = \frac{2^3}{3^2}$

$\Rightarrow f^{(3)}(1) = \frac{8}{9} \cdot 6 = \frac{16}{3}$

21) Which of the following is a power series representation of the function

$$f(x) = \frac{x-1}{x^2-2x+10}?$$

A) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n+1}}{9^{n+1}}$

B) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{2n+1}}{9^{n+1}}$

C) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n+1}}{3^{n+1}}$

D) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{2n+2}}{9^n}$

E) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^n}{3^{n+1}}$

$$f(x) = \frac{x-1}{(x^2-2x+1)+9} = (x-1)\left(\frac{1}{9+(x-1)^2}\right)$$

$$= \frac{(x-1)}{9}\left(\frac{1}{1-\left(-\left(\frac{x-1}{3}\right)^2\right)}\right)$$

$$= \frac{x-1}{9}\sum_{n=0}^{\infty}\left(-\left(\frac{x-1}{3}\right)^2\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{9^{n+1}}(x-1)^{2n+1}$$

22) The foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ are

A) $(-3,0)$ and $(3,0)$

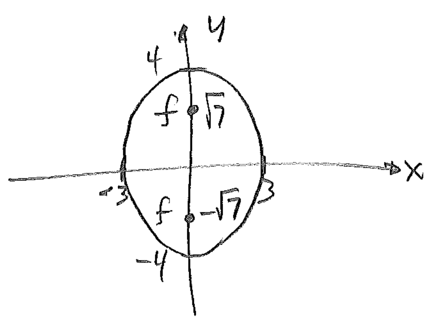
B) $(-5,0)$ and $(5,0)$

C) $(0,-\sqrt{7})$ and $(0,\sqrt{7})$

D) $(-\sqrt{7},0)$ and $(\sqrt{7},0)$

E) $(0,-3)$ and $(0,3)$

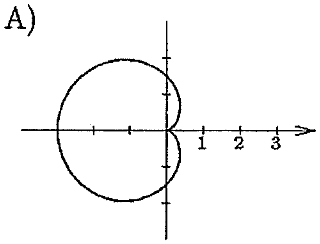
$$c^2 = 16-9 = 7 \rightarrow c = \pm\sqrt{7}$$



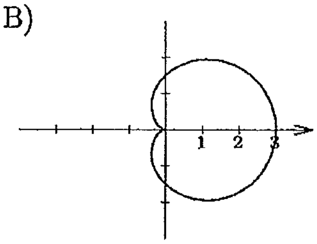
(major axis is along y-axis)

23) The graph of the curve given by the equation $r = 1 - 2\cos\theta$ looks mostly like

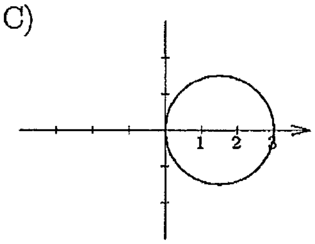
A)



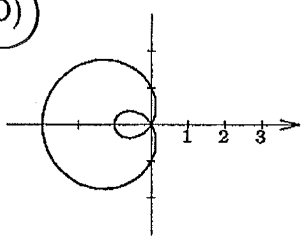
B)



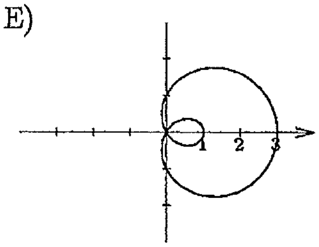
C)



D)



E)



θ	0	$\pi/2$	π	$3\pi/2$
$r = 1 - 2\cos\theta$	-1	1	3	1

24) Which of the following are polar coordinates of the point whose Cartesian coordinates are $(-1, -\sqrt{3})$?

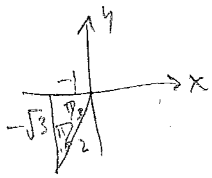
A) $r = 1$, $\theta = \dfrac{\pi}{3}$.

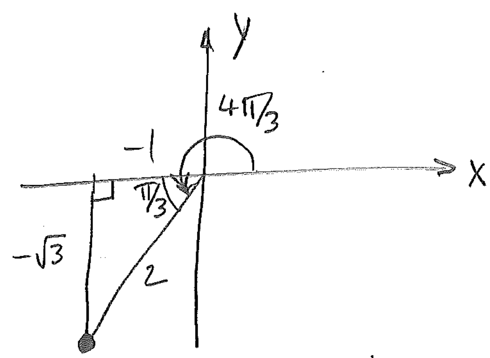
B) $r = 2$, $\theta = \dfrac{2\pi}{3}$

C) $r = 2$, $\theta = \dfrac{7\pi}{6}$

D) $r = 2$, $\theta = \dfrac{4\pi}{3}$

E) $r = 2$, $\theta = \dfrac{7\pi}{6}$





$(r, \theta) = (2, \frac{4\pi}{3})$

25) The complex number $\dfrac{1+3i}{3+4i}$ is equal to

A) $7+\frac{2}{3}i$

B) $\frac{2}{3}+\frac{1}{3}i$

C) $\frac{3}{5}+\frac{1}{5}i$

D) $\frac{2}{5}+\frac{3}{5}i$

E) $\frac{3}{7}+\frac{1}{7}i$

$$\frac{1+3i}{3+4i}\cdot\frac{3-4i}{3-4i}$$

$$=\frac{3+5i-12i^2}{9-16i^2}$$

$$=\frac{3+5i+12}{9+16}$$

$$=\frac{15+5i}{25}$$

$$=\frac{15}{25}+\frac{5}{25}i$$

$$=\frac{3}{5}+\frac{1}{5}i$$