MATH 162 – SPRING 2010 – THIRD EXAM – APRIL 13, 2010

VERSION 01

MARK TEST NUMBER 01 ON YOUR SCANTRON

STUDENT NAME——SOLUTIONS——

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INSTRUCTIONS

1. Fill in all the information requested above and the version number of the test on your scantron sheet.
2. This booklet contains 13 problems. Problem 1 is worth 4 points. The others are worth 8 points each. The maximum score is 100 points.
3. For each problem mark your answer on the scantron sheet and also circle it is this booklet.
4. Work only on the pages of this booklet.
5. Books, notes and calculators are not allowed.
6. At the end turn in your exam and scantron sheet to your recitation instructor.

Useful Formulas

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

1)(4 points) For the series $\sum_{n=1}^{\infty} (-1)^n n^2$, the partial sum s_4 equals

A) 2.

B) 10.

C) -10.

D) -2.

E) 30.

$$s_4 = \sum_{n=1}^{4} (-1)^n n^2 = -1^2 + 2^2 - 3^2 + 4^2$$
$$= -1 + 4 - 9 + 16$$
$$= 10$$

2)(8 points) Which of the following statements are true?

(I) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. False. $\sum^{\infty} \frac{1}{n}$ diverges

(II) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. True. Absolute convergence $\Rightarrow$ Convergence

(III) If $\sum_{n=1}^{\infty} \left|\frac{a_{n+1}}{a_n}\right|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. True. $\sum^{\infty} \left|\frac{a_{n+1}}{a_n}\right|$ converges $\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \Rightarrow \sum^{\infty} a_n$ converges by Ratio Test.

(IV) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. False. $\frac{1}{n^2} < \frac{1}{n}$. $\sum^{\infty} \frac{1}{n}$ diverges but $\sum^{\infty} \frac{1}{n^2}$ converges

(V) If $\lim_{n\to\infty} 5^n a_n = 2$, then $\sum_{n=1}^{\infty} a_n$ converges. True.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{5}\right)^n}{a_n} = \lim_{n\to\infty} \frac{1}{5^n a_n} = \frac{1}{2} > 0 \text{ and } \sum^{\infty} \left(\frac{1}{5}\right)^n \text{ converges.}$$

$\Rightarrow \sum^{\infty} a_n$ converges by Limit Comparison Test.

A)(I), (II) and (III) only.

B)(I), (II) and (IV) only.

C)(II), (IV) and (V) only.

D)(II), (III) and (V) only.

E)(II), (III) and (IV) only.

3)(8 points) Which of the following alternatives is true about the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$?

A) It converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.

B) It diverges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.

C) It converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

D) It diverges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

E) It converges by the integral test.

$$\int_2^{\infty} (\log x)^{-2} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_2^t (\log x)^{-2} \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\frac{-1}{\log x} \Big|_2^t \right) = \lim_{t\to\infty} \left(-\frac{1}{\log t} + \frac{1}{\log 2} \right) = 0 + \frac{1}{\log 2}.$$

4)(8 points) Which of the following series diverge?

(I) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ DIV. $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n+1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1 > 0$ and $\sum^{\infty} \frac{1}{n}$ DIV.

Limit Comparison Test

(II) $\sum_{n=1}^{\infty} \frac{n^2+n}{n^2-n}$ DIV. $\lim_{n\to\infty} \frac{n^2+n}{n^2-n} = 1 \neq 0$. The Divergence Test

(III) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ DIV. $\ln n < n$ for $n \geq 1$.

$\rightarrow \frac{1}{\ln n} > \frac{1}{n}$ and $\sum^{\infty} \frac{1}{n}$ diverges.

Comparison Test

A)(I) only.

B)(II) only.

C)(I) and (II) only.

D)(II) and (III) only.

E)All of them.

5)(8 points) Which statement is true about the following series?

(I) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conditionally convergent. $\sum^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv (alt. series test) but $\sum^{\infty} \frac{1}{\sqrt{n}}$ div., p-series, $p=\frac{1}{2}<1$

(II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely convergent $\sum^{\infty} \frac{1}{n^2}$ converges, p-series, $p=2>1$

(III) $\sum_{n=1}^{\infty} (-1)^n\sqrt{n}$ diverges $\lim_{n\to\infty} \sqrt{n} \neq 0$. The divergence test.

A)All are conditionally convergent.

B)All are divergent.

C)(I) is conditionally convergent; (II) is absolutely convergent.

D)(I) is absolutely convergent; (II) is conditionally convergent.

E)(I) and (II) are conditionally convergent; (III) is divergent.

6)(8 points) Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n-1)}$. Find the smallest integer N such that we can be sure that $|S_N - S| < \frac{1}{100}$, where $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(n+1)(n-1)}$

A) 8.

B) 9.

C) 10.

D) 11.

E) 12.

n	$\frac{1}{(n+1)(n-1)}$
8	$\frac{1}{(9)(7)} = \frac{1}{63} > \frac{1}{100}$
9	$\frac{1}{(10)(8)} = \frac{1}{80} > \frac{1}{100}$
10	$\frac{1}{(11)(9)} = \frac{1}{99} > \frac{1}{100}$
11	$\frac{1}{(12)(10)} = \frac{1}{120} < \frac{1}{100} \rightarrow \lvert a_{11}\rvert < \frac{1}{100}$

$\Rightarrow$ Therefore smallest $N = 10$

* therefore interval of convergence is $(1, 3]$ and radius of convergence is 1.

7)(8 points) The radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{(n+1)}$ satisfy

A) The radius is equal to 1 and the interval is $(-1, 1)$.

B) The radius is equal to 2 and the interval is $(0, 4)$.

C) The radius is equal to 1 and the interval is $(1, 3)$.

D) The radius is equal to 1 and the interval is $(1, 3]$.

E) The radius is equal to 1 and the interval is $[1, 3]$.

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n(x-2)^n}\right|$

$= \lim_{n\to\infty}\frac{n+1}{n+2}|x-2| = |x-2|.$

Series converges for $|x-2|<1$
$\longrightarrow -1 < x-2 < 1 \longrightarrow 1 < x < 3$

Endpts: $x=1 \Rightarrow \sum^{\infty}\frac{(-1)^n(-1)^n}{n+1} = \sum^{\infty}\frac{1}{n+1}$ which diverges. (Limit comparison with $\sum^{\infty}\frac{1}{n}$)

$x=3 \Rightarrow \sum^{\infty}\frac{(-1)^n(1)^n}{n+1}$ which converges (alt. series test) * (see above)

8)(8 points) Which of the following is a power series representation of the function $f(x) = \frac{x-2}{x^2-4x+5}$?

A) $\sum_{n=0}^{\infty}\frac{1}{n!}(x-2)^n$.

B) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(x-2)^n$.

C) $\sum_{n=0}^{\infty}(x-2)^{n+1}$.

D) $\sum_{n=0}^{\infty}(-1)^n(x-2)^{\cancel{n+1}\ 2n+1}$.

E) $\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}(x-2)^{n+1}$.

$\frac{x-2}{x^2-4x+5} = \frac{x-2}{(x^2-4x+4)+1} = \frac{x-2}{1-(-(x-2)^2)}$

$=(x-2)\sum_{n=0}^{\infty}\left(-(x-2)^2\right)^n = (x-2)\sum_{n=0}^{\infty}(-1)^n(x-2)^{2n}$

$= \sum_{n=0}^{\infty}(-1)^n(x-2)^{2n+1}$

9)(8 points) The Maclaurin series of the functon $f(x) = \dfrac{1}{(4-x)^3}$ is
(Hint: Start with the power series of $(4-x)^{-1}$ and differentiate it enough times.)

A) $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2(4^{n+1})} x^{n-2}$.

B) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{4^n} x^{n-2}$.

C) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{4^n} x^{n-2}$.

D) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{4^{n+2}} x^{n-2}$.

E) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n^2(n-1)}{2(4^n)} x^{n-2}$.

$$\frac{1}{4-x} = \frac{1}{4}\left(\frac{1}{1-\left(\frac{x}{4}\right)}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}} x^n$$

$$\frac{d}{dx}\left(\frac{1}{4-x}\right) = \frac{1}{(4-x)^2} = \sum_{n=1}^{\infty}\frac{n}{4^{n+1}} x^{n-1}$$

$$\frac{d}{dx}\left(\frac{-1}{(4-x)^2}\right) = \frac{2}{(4-x)^3} = \sum_{n=2}^{\infty}\frac{(n)(n-1)}{4^{n+1}} x^{n-2}$$

$$\Rightarrow f(x) = \frac{1}{(4-x)^3} = \sum_{n=2}^{\infty}\frac{(n)(n+1)}{2(4^{n+1})} x^{n-2}$$

10)(8 points) The Maclaurin series of $f(x) = (\cos x)^2$ is equal to
(Hint: Use that $(\cos x)^2 = \frac{1}{2}(1+\cos 2x)$.)

A) $\displaystyle\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$.

B) $\displaystyle\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{2(2n)!} x^{2n}$.

C) $\displaystyle\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 8^n}{4(n!)} x^{2n}$.

D) $\displaystyle\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{2(2n)!} x^n$.

E) $\displaystyle\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{4n}$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

$$\frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{2(2n)!} x^{2n}$$

11)(8 points) Let $f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-2)^n$. We can say that the fifth derivative of f at the point 2 is equal to

$$\frac{f^{(5)}(2)}{5!} = \frac{2^5}{5!} \to f^{(2)}(2) = 2^5 = 32$$

A) $f^{(5)}(2) = 10.$

B) $f^{(5)}(2) = 64.$

C) $f^{(5)}(2) = 32.$

D) $f^{(5)}(2) = 21.$

E) $f^{(5)}(2) = 100.$

$$\left(\text{note: } \frac{f^{(n)}(2)}{n!} = \frac{2^n}{n!} . \text{ Let } n=5. \right)$$

12)(8 points) If we use that $\frac{1}{\sqrt{1-x}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\ n!} x^n$, and that $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$, we conclude that the Maclaurin series of $\arcsin x$ is equal to

A) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (2n+1)\ n!} x^{2n+1}.$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\ n!} x^{2n}$$

B) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+3} (2n+1)!} x^{2n+1}.$

C) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (2n+1) n!} x^n.$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n+1)(2^n)(n!)} x^{2n+1}$$

D) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\ n!} x^{2n+1}.$

E) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\ n!} x^{2n+3}.$

13)(8 points) Let $f(x)$ be a function defined on $[1, \infty)$ such that $f(x) > 1$ for all x and $\lim_{x \to \infty} \frac{f(x)}{x} = 1$. What can we say about the convergence of the series

$$S_1 = \sum_{n=1}^{\infty} \sin\left(\frac{1}{f(n)}\right) \text{ and } S_2 = \sum_{n=1}^{\infty} \sin\left(\frac{1}{f(n)^3}\right)?$$

A) S_1 and S_2 diverge.

B) S_1 converges and S_2 diverges.

C) S_1 diverges and S_2 converges.

D) S_1 and S_2 converge.

E) Nothing can be said about the convergence of the series.

<u>note</u> $\lim_{x \to \infty} \frac{f(x)}{x} = 1$ implies $\lim_{x \to \infty} f(x) = \infty$

$$\lim_{x \to \infty} \frac{f(x)}{x} = 1 \Rightarrow \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{f(x)}} = 1$$

$$\text{and } \lim_{x \to \infty} \left[\frac{\frac{1}{x}}{\frac{1}{f(x)}}\right]^3 = 1$$

Consequently $\sum^{\infty} \frac{1}{f(n)}$ diverges since $\sum^{\infty} \frac{1}{n}$ diverges.

and $\sum^{\infty} \frac{1}{(f(n))^3}$ converges since $\sum^{\infty} \frac{1}{n^3}$ converges.

Using the Limit Comparison Test again,

$$\lim_{n \to \infty} \frac{\sin \frac{1}{f(n)}}{\frac{1}{f(n)}} = 1 \qquad \lim_{n \to \infty} \frac{\sin\left(\frac{1}{(f(n))^3}\right)}{\frac{1}{(f(n))^3}} = 1$$

so $\sum^{\infty} \sin\left(\frac{1}{f(x)}\right)$ diverges

So $\sum^{\infty} \sin\left(\frac{1}{(f(n))^3}\right)$ converges