SOLUTIONS

MA 16200 Exam 3 Spring 2009

1. Which of the following series converges?

(I) $\sum_{n=1}^{\infty} \frac{2}{n^{0.99}}$ diverges $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$ is a p-series, $p = 0.99 < 1$ $\rightarrow$ series diverges

(II) $\sum_{n=1}^{\infty} \frac{1-2\sqrt{n}}{n^2}$ (absolutely) convergent

Limit Comparison Test with convergent $\sum^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\left|\frac{1-2\sqrt{n}}{n^2}\right|}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{2n^2 - n^{3/2}}{n^2} = 2 > 0.$$

Thus $\sum^{\infty} \frac{2\sqrt{n}-1}{n^2}$ converges, so $\sum^{\infty} \frac{1-2\sqrt{n}}{n^2}$ converges absolutely

(III) $\sum_{n=1}^{\infty} \frac{1-2\sqrt{n}}{n^{3/2}}$ diverges

Limit Comparison Test with divergent $\sum^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{2\sqrt{n}-1}{n^{3/2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^{3/2} - n}{n^{3/2}} = 2 > 0.$$

Thus $\sum^{\infty} \frac{2\sqrt{n}-1}{n^{3/2}}$ diverges, so $-\sum^{\infty} \frac{2\sqrt{n}-1}{n^{3/2}}$ diverges

A. All of them.

B. (II) only.

C. (I) and (II) only.

D. (II) and (III) only.

E. (I) only.

2. Which of the following statements is true?

(I) If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. TRUE (Comparison Test)

(II) If $a_n \ge b_n \ge 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. TRUE (Comparison Test)

(III) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. False $\frac{1}{n^2} < \frac{1}{n}$, $n > 1$ and $\sum^{\infty} \frac{1}{n^2}$ converges and $\sum^{\infty} \frac{1}{n}$ diverges

A. (I) only.

B. (II) only.

C. (I) and (III) only.

D. (II) and (III) only.

E. (I) and (II) only.

3. Which of the following series converges?

(I) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+1}}}{\frac{1}{n^{3/2}}} = 1 > 0$ and $\sum^{\infty} \frac{1}{n^{3/2}}$ converges

(II) $\sum_{n=1}^{\infty} \frac{n+3^n}{n+5^n}$ converges $\lim_{n\to\infty} \frac{\frac{n+3^n}{n+5^n}}{\left(\frac{3}{5}\right)^n} = \lim_{n\to\infty}\left(\frac{n+3^n}{n+5^n}\right)\left/\left(\frac{\left(\frac{1}{3}\right)^n}{\left(\frac{1}{5}\right)^n}\right)\right. = 1 > 0$

and $\sum^{\infty}\left(\frac{3}{5}\right)^n$ converges

(III) $\sum_{n=1}^{\infty} \frac{n+3}{(n+2)^3}$ converges

A. All of them.
B. (I) and (II) only.
C. (I) and (III) only.
D. (II) and (III) only.
E. (I) only.

$\lim_{n\to\infty} \frac{\frac{n+3}{(n+2)^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+3}{n^3+6n^2+6n+8} = 1 > 0$

and $\sum^{\infty} \frac{1}{n^2}$ converges.

4. Which of the following statements is correct (only one of them is correct):

TRUE A. The series $\sum_{n=1}^{\infty} \sin\frac{0.1}{n^2}$ converges, by the limit comparison test. $\lim_{n\to\infty} \frac{\sin\left(\frac{0.1}{n^2}\right)}{\frac{0.1}{n^2}} = 1 > 0$ $\left(\frac{0.1}{n^2}\to 0\right)$

$\sum^{\infty} \frac{0.1}{n^2}$ converges

FALSE B. The series $\sum_{n=1}^{\infty} \sin\frac{0.1}{n^2}$ converges, because $\sin\frac{0.1}{n^2} \to 0$, as $n\to\infty$.

No series converges simply because $\lim_{n\to\infty} a_n = 0$.

FALSE C. The series $\sum_{n=1}^{\infty} \sin\frac{0.1}{n^2}$ is an alternating series, and therefore is convergent.

It is not an alternating series, and not all alternating series converge.

FALSE D. The series $\sum_{n=1}^{\infty} \sin\frac{0.1}{n^2}$ diverges by the ratio test.

$\lim_{n\to\infty} \frac{\sin\frac{0.1}{(n+1)^2}}{\sin\frac{0.1}{n^2}} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{\left(\cos\frac{0.1}{(n+1)^2}\right)\left(\frac{-0.2}{(n+1)^3}\right)}{\left(\cos\frac{0.1}{n^2}\right)\left(\frac{-0.2}{n^3}\right)} = 1 \to$ Ratio Test Inconclusive

FALSE E. $\sum_{n=1}^{\infty} \sin\frac{0.1}{n^2}$ diverges, because the integral $\int_1^{\infty} \sin x\, dx$ is divergent.

$\sin x \neq \sin\frac{0.1}{x^2}$. Wrong improper integral.

5. For the series

(I) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ Cond. Conv. $\left\{ \begin{array}{l} \sum^{\infty} \frac{(-1)^n}{\ln n} \text{ is a convergent alternating series} \\ \sum^{\infty} \frac{1}{\ln n} \text{ diverges. } \underline{\text{note}}: \frac{1}{\ln n} > \frac{1}{n} \text{ and } \sum^{p} \frac{1}{n} \text{ diverges.} \end{array} \right.$

(II) $\sum_{n=1}^{\infty} (-1)^n \frac{e^{\frac{1}{n}}}{n^3}$ abs. conv. $\left\{ \begin{array}{l} e^{1/n} \le e \text{ for } n \geq 1 \Rightarrow \frac{e^{1/n}}{n^3} \le \frac{e}{n^3} \text{ and } \sum^{\infty} \frac{e}{n^3} \text{ converges.} \\ \underline{\text{note}}: 0 < \frac{1}{n} < 1 \rightarrow e^0 < e^{1/n} < e^1 \rightarrow 1 < e^{1/n} < e \end{array} \right.$

A. (I) and (II) are absolutely convergent.

B. (I) is divergent, (II) is absolutely convergent.

C. (I) is conditionally convergent, (II) is absolutely convergent.

D. (I) and (II)are conditionally convergent.

E. (I) is divergent, (II) is conditionally convergent.

6. The series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

A. Converges absolutely by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^n}$.

B. Diverges since $\lim_{n\to\infty} \frac{(-2)^n}{n^n} \neq 0$.

C. Converges absolutely by the root test.

D. Diverges by the ratio test.

E. Diverges by the root test.

$\lim_{n\to\infty} \left| \frac{(-2)^n}{n^n} \right|^{1/n}$

$= \lim_{n\to\infty} \frac{2}{n} = 0 < 1$

$\longrightarrow$ series converges absolutely by the Root Test.

7. Consider the following series:

I. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ diverges since $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$.

II. $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ converges since $\frac{1}{n+3^n} < \frac{1}{3^n}$ and $\sum^{\infty} \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ converges

III. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ converges by alt. series test

$\lim_{n\to\infty} \frac{n}{n^2+2} = 0$ and $f(x) = \frac{x}{x^2+2} \to f'(x) = \frac{(1)(x^2+2)-(x)(2x)}{(x^2+2)^2}$

$= \frac{2-2x^2}{(x^2+2)^2} < 0$ for $2-2x^2<0$

$\to 2 < 2x^2$

$\to x^2 > 1$

$\to x > 1$.

A. They all converge.

B. Only (I) and (II) converge.

C. Only (I) and (III) converge.

D. Only (II) and (III) converge.

E. They all diverge.

8. Consider the following series:

conv. I. $\sum_{n=1}^{\infty} \frac{n^2}{n+1}\frac{1}{2^n}$ Ratio test: $\lim_{n\to\infty} \frac{(n+1)^2}{n+2}\frac{1}{2^{n+1}} \cdot \frac{(n+1)\,2^n}{n^2} = \lim_{n\to\infty} \frac{(n^3+3n^2+3n+1)}{(n^3+2n^2)\cdot 2} = \frac{1}{2}$ and $\frac{1}{2} < 1$.

conv. II. $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ Alt Series Test: $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$ and $f(x) = \frac{1}{x} + \frac{1}{x^2} \to f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$

$\to f'(x) = \frac{-x-2}{x^3} < 0 \to -x-2 < 0 \to x > -2$.

div. III. $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ Limit Comparison Test: $\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1 > 0$ and $\sum^{\infty} \frac{1}{n}$ diverges.

A. They all converge.

B. Only (I) and (II) converge.

C. Only (I) and (III) converge.

D. Only (II) and (III) converge.

E. They all diverge.

9. Find the radius of convergence of $\sum_{n=0}^{\infty} \sqrt{n}2^n x^n$.

A. 0

B. $\frac{1}{2}$

C. 1

D. 2

E. ∞

Root Test: $\lim_{n\to\infty} \left| \sqrt{n}\, 2^n x^n \right|^{1/n} = \lim_{n\to\infty} (n^{1/n})^{1/2} (2)|x|$

$= 2|x|$. Series converges for $2|x| < 1$

$\longrightarrow |x| < \frac{1}{2} \rightarrow -\frac{1}{2} < x < \frac{1}{2}$.

$\Rightarrow$ Radius of convergence is $\frac{1}{2}$.

10. Given that the radius of convergence of $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ is 1, find the interval of convergence.

A. $(2, 4)$

B. $[2, 4]$

C. $[2, 4)$

D. $(2, 4]$

E. None of the above.

radius of convergence is 1

$\Rightarrow$ series converges for $|x-3| < 1$

$\rightarrow -1 < x-3 < 1$

$\rightarrow 2 < x < 4$

$x = 2 \Rightarrow \sum^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum^{\infty} \frac{1}{2n+1}$ divergent (limit comparison with $\sum^{\infty} \frac{1}{n}$)

$x = 4 \rightarrow \sum^{\infty} (-1)^n \frac{1}{2n+1}$ convergent (alt. series test.)

11. Find a power series for the indefinite integral $F(t) = \int \frac{t}{1-t^8}\,dt$ and find its radius of convergence R.

A. $F(t) = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$, $R = 1$

B. $F(t) = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$, $R = \infty$

C. $F(t) = C + \sum_{n=0}^{\infty} t^{8n+1}$, $R = 1$

D. $F(t) = C + \sum_{n=0}^{\infty} t^{8n+1}$, $R = \infty$

E. None of the above.

$$\frac{t}{1+t^8} = \sum_{n=0}^{\infty} t(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1}, \quad |t|<1.$$

$$\int \frac{t}{1-t^8}\,dt = \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C,$$

$|t|<1$

$\rightarrow R=1$

12. Find the Taylor series for $f(x) = e^{2x}$ centered at $a = 3$.

A. $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

B. $\sum_{n=0}^{\infty} \frac{2^n}{n!} e^3 (x+3)^n$

C. $\sum_{n=0}^{\infty} \frac{2^n}{n!} e^6 (x+3)^n$

D. $\sum_{n=0}^{\infty} \frac{2^n}{n!} e^3 (x-3)^n$

E. $\sum_{n=0}^{\infty} \frac{2^n}{n!} e^6 (x-3)^n$

$f(x) = e^{2x}$ $\qquad f(3) = e^6$

$f'(x) = 2e^{2x}$ $\qquad f'(3) = 2e^6$

$f''(x) = 2^2 e^{2x}$ $\qquad f''(3) = 2^2 e^6$

$\vdots$ $\qquad \vdots$

$f^{(n)}(x) = 2^n e^{2x}$ $\qquad f^{(n)}(3) = 2^n e^6$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n$$