MATH 162 – SPRING 2004 – THIRD EXAM
SOLUTIONS

Useful formulas:

Arc length

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \; dt$$

Area of a surface of revolution

$$S = \int_a^b 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

Some power series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{provided} \quad |x| < 1$$

1) Find which series equals the definite integral $\int_0^1 \sin(x^2)\, dx$

A) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+2)!}$

B) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+3)!}$

C) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(4n+3)}$

D) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{(2n+5!}$

E) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(4n+2)}$

Solution: Using the formula given above for the Maclaurin series of $\sin x$, but with x replaced by x^2, we have

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

Therefore

$$\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 \frac{x^{4n+2}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(4n+3)(2n+1)!}.$$

The correct answer is C.

2) The power series expansion of $\frac{1}{(1+x)^2}$ is

A) $\sum_{n=0}^{\infty}(-1)^n x^n$

B) $\sum_{n=0}^{\infty}(-1)^n n x^{n-1}$

C) $\sum_{n=0}^{\infty}(-1)^{n-1} n x^{n-1}$

D) $\sum_{n=0}^{\infty}(-1)^{n-1} x^n$

E) $\sum_{n=0}^{\infty} x^n$

Solution: We know that

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$$

and by the formula given above we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Therefore

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty} n(-1)^n x^{n-1}.$$

Notice that the term corresponding to $n = 0$ is zero. One could also state that

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-1)^n x^{n-1}.$$

The correct answer is B.

3) If $(1+x)^{1/3} = c_1 + c_2 x + c_3 x^2 + ...$ then c_3 is equal to

A) $\frac{1}{3}$

B) $\frac{1}{5}$

C) $\frac{1}{9}$

D) $\frac{1}{12}$

E) $-\frac{1}{9}$

Solution: The binomial theorem says that for any k real

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)...(k-n+1)}{n!} \quad x^n.$$

So the term in x^2 is $\frac{k(k-1)}{2}$. In this case $k = \frac{1}{3}$ so $c_3 = \frac{\frac{1}{3}(\frac{1}{3}-1)}{2} = -\frac{1}{9}$. The correct answer is E.

4) The MacLaurin series of $x\cos(2x)$ is

A) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$

B) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+1}}{(2n)!}$

C) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n}}{(2n)!}$

D) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+1}}{(2n)!}$

E) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n+1}}{(2n)!}$

Solution: Using the formula provided above for the MacLaurin series of $\cos x$, but with x replaced by $2x$ we have

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n} x^{2n}}{(2n)!}$$

Multiplying this by x gives

$$x\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n} x^{2n+1}}{(2n)!}.$$

The correct answer is should have been C, however, as you can see, due to a typo the question had no soution. Everyone was given 10 points for this question.

5) The Taylor polynomial $T_2(x)$ for $f(x) = \sin x$ at $a = \frac{\pi}{3}$ is

A) $\frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{3}}{4}(x - \frac{\pi}{3})^2$

B) $\frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) + \frac{\sqrt{3}}{4}(x - \frac{\pi}{3})^2$

C) $\frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{4}(x - \frac{\pi}{3})^2$

D) $\frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) + \frac{1}{4}(x - \frac{\pi}{3})^2$

E) $(x - \frac{\pi}{3}) - \frac{1}{6}(x - \frac{\pi}{3})^2$

Solution: We know that the Taylor polynomial of degree n of a function f at a point $x = a$ is

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x - a)^j.$$

Here we have $f(x) = \sin x$, $n = 2$ and $a = \frac{\pi}{3}$.

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x.$$

Evaluating these at $x = \frac{\pi}{3}$ gives

$$f(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \sqrt{3}2, \quad f'(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}, \quad f''(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\sqrt{3}2.$$

So

$$T_2(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{3}}{4}(x - \frac{\pi}{3})^2.$$

The correct answer is A.

6) The slope of the tangent line to the graph of the curve $x = 1 + t^2$, $y = t \ln t$ at $t = 2$ is

A) $\frac{1}{4}$

B) $\frac{\ln 2}{4}$

C) $\frac{4}{\ln 2}$

D) $\frac{1+\ln 2}{4}$

E) $\frac{4}{1+\ln 2}$

Solution: We know that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln t + 1}{2t}.$$

When $t = 2$ we have

$$\frac{dy}{dx} = \frac{1 + \ln 2}{4}.$$

The correct answer is D.

7) The length of the curve $x = e^t + e^{-t}$, $y = 2t$, $0 \leq t \leq 1$ is

A) $e + e^{-1} - 2$

B) $e - e^{-1}$

C) $e + e^{-1}$

D) $e + e^{-1} - 2$

E) $\frac{1}{2}(e + e^{-1})$

Solution: First we find that

$$x'(t) = e^t - e^{-t}, \quad y'(t) = 2$$

So

$$(x'(t))^2 + (y'(t))^2 = (e^t + e^{-t})^2 + 4 = e^{2t} - 2 + e^{2t} + 4 = e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2.$$

So

$$L = \int_0^1 (e^t + e^{-t})\, dt = e^t - e^{-t}\big|_0^1 = e - e^{-1}.$$

The correct answer is C.

8) The curve $x = \cos^3\theta$, $y = \sin^3\theta$, $0 \le \theta \le \frac{\pi}{2}$ is rotated about the x-axis to generate a surface. Its area is given by

A) $\int_0^{\frac{\pi}{2}} 6\pi \cos\theta \sin\theta \, d\theta$

B) $\int_0^{\frac{\pi}{2}} 6\pi \cos^2\theta \sin^2\theta \, d\theta$

C) $\int_0^{\frac{\pi}{2}} 6\pi \cos^2\theta \sin^3\theta \, d\theta$

D) $\int_0^{\frac{\pi}{2}} 6\pi \cos\theta \sin^4\theta \, d\theta$

E) $\int_0^{\frac{\pi}{2}} 6\pi \cos^2\theta \sin^4\theta \, d\theta$

Solution: We find that

$$x'(\theta) = -3\cos^2\theta \sin\theta, \quad y'(\theta) = 3\sin^2\theta\cos\theta.$$

So

$$(x'(\theta))^2 + (y'(\theta))^2 = 9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta = 9\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) = 9\sin^2\theta\cos^2\theta.$$

Therefore

$$\sqrt{(x'(\theta))^2 + (y'(\theta))^2} = 3\cos\theta\sin\theta$$

So finally

$$A = 2\pi \int_0^{\frac{\pi}{2}} y(\theta)\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = 6\pi \int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta \, d\theta.$$

The correct answer is D.

9) The cartesian coordinates of a point are $(-2\sqrt{3}, 2)$. Find its polar coordinates

A) $(4, \frac{2\pi}{3})$

B) $(4, \frac{5\pi}{6})$

C) $(2, \frac{2\pi}{3})$

D) $(2, \frac{5\pi}{6})$

E) $(4, -\frac{\pi}{3})$

Solution: We know that $x = r\cos\theta$ and $y = r\sin\theta$ where $r^2 = x^2 + y^2$. So $r^2 =$

$(-2\sqrt{3})^2 + 4 = 16$. Then $r = 4$. On the other hand

$$\cos\theta = \frac{x}{r} = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{y}{r} = \frac{1}{2}.$$

The angle must be on the second quadrant and so $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The correct answer is B.

10) The polar equation of the circle of radius 1 centered at $(0, -1)$ is

A) $r = 2\cos\theta$

B) $r = 2\sin\theta$

C) $r = -\sin\theta$

D) $r = -2\sin\theta$

E) $r = -2\cos\theta$

Solution: The circle centered at $(0, -1)$ with radius 1 has equation $x^2 + (y+1)^2 = 1$. Then $x^2 + y^2 + 2y + 1 = 1$ and thus $x^2 + y^2 + 2y = 0$. Since $x^2 + y^2 = r^2$ and $y = r\sin\theta$, this equation reduces to $r^2 + 2r\sin\theta = 0$ or $r = -2\sin\theta$.