MATH 162 – SPRING 2010 – SECOND EXAM – MARCH 9, 2010

VERSION 01

MARK TEST NUMBER 01 ON YOUR SCANTRON

STUDENT NAME—Solution Key

STUDENT ID—

RECITATION INSTRUCTOR—

INSTRUCTOR—

RECITATION TIME—

INSTRUCTIONS

1. Fill in all the information requested above and the version number of the test on your scantron sheet.

2. This booklet contains 12 problems, each worth 8 points. There are four free points. The maximum score is 100 points.

3. For each problem mark your answer on the scantron sheet and also circle it is this booklet.

4. Work only on the pages of this booklet.

5. Books, notes and calculators are not allowed.

6. At the end turn in your exam and scantron sheet to your recitation instructor.

USEFUL INTEGRALS

$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{a^2+u^2}) + C$$

$$\int \sqrt{u^2+1}\, du = \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2}) + C$$

1) What is the most suitable substitution to calculate the integral

$$\int \frac{\sqrt{9x^2-4}}{x}\,dx?$$

A) $3x = 2\sin\theta$

B) $3x = 2\tan\theta$

C) $3x = 2\sec\theta$

D) $2x = 3\tan\theta$

E) $2x = 3\sin\theta$

$$\sqrt{9x^2-4} = \sqrt{(3x)^2-(2)^2}$$

$\Rightarrow$ Let $3x = 2\sec\theta$

2) Which of the following integrals do you get when you make a suitable trigonometric substitution to evaluate

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx?$$

A) $\int \frac{\sin^3\theta}{\cos\theta}\,d\theta$

B) $\int \tan^3\theta\sec\theta\,d\theta$

C) $\int \frac{\tan^3\theta}{\sec\theta}\,d\theta$

D) $\int \sec^4\theta\,d\theta$

E) $\int \sin^3\theta\,d\theta$

Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$,
and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$
(for suitable choices of θ)

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \int \sin^3\theta\,d\theta$$

3) Evaluate the integral

$$\int_0^2 \frac{x^2}{(x^2+4)^2}\,dx. = I$$

A) $\frac{\pi}{16} - \frac{1}{8}$

B) $\frac{\pi}{4} + \frac{1}{3}$

C) $\frac{\pi}{8} - \frac{1}{8}$

D) $\frac{\pi}{12} - \frac{1}{4}$

E) $\frac{\pi}{9} + \frac{1}{8}$

Let $x = 2\tan\theta$. $dx = 2\sec^2\theta\, d\theta$. $(x^2+4)^2 = 16\sec^4\theta$

$\theta = \tan^{-1}\frac{x}{2}$. $\theta(0) = \tan^{-1} 0 = 0$. $\theta(2) = \tan^{-1} 1 = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \frac{4\tan^2\theta}{16\sec^4\theta}\sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(\frac{\tan^2\theta}{\sec^2\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right) d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1 - \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)\right) d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{4} - \frac{\cos 2\theta}{4}\right) d\theta$$

$$= \left(\frac{1}{4}\theta - \frac{\sin 2\theta}{8}\right)\Big|_0^{\frac{\pi}{4}} = \left(\frac{\pi}{16} - \frac{1}{8}\right) - (0-0).$$

4) The form of the partial fraction decomposition of $\frac{17x-3}{x^4-16}$ is

A) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$

B) $\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$

C) $\frac{A}{x-4} + \frac{B}{x+4} + \frac{Cx+D}{x^2+4}$

D) $\frac{A}{x^2-4} + \frac{B}{x^2+4}$

E) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x^2+4}$

$$\frac{17x-3}{x^4-16} = \frac{17x-3}{(x^2-4)(x^2+4)} = \frac{17x-3}{(x-2)(x+2)(x^2+4)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

5) Evaluate the integral

$$\int_2^3 \frac{2x}{(1+x)(x-1)}\,dx.$$

A) 1/2

B) ln(2/5)

C) ln(4/3)

D) $2\ln 3$

E) ln(8/3)

$$\frac{2x}{(1+x)(x-1)} = \frac{A}{1+x} + \frac{B}{x-1}$$

$$\longrightarrow \quad 2x = A(x-1) + B(1+x)$$

$$x=1 \longrightarrow 2 = 0\cdot A + 2\cdot B \longrightarrow B=1$$

$$x=-1 \longrightarrow -2 = -2\cdot A + 0\cdot B \longrightarrow A=1$$

$$\int_2^3 \left(\frac{1}{1+x} + \frac{1}{x-1}\right)dx = \left(\ln|1+x| + \ln|x-1|\right)\Big|_2^3$$

$$= (\ln 4 + \ln 2) - (\ln 3 + \cancel{\ln 1}^{\,0})$$

$$= \ln\left(\frac{4\cdot 2}{3}\right) = \ln\left(\frac{8}{3}\right)$$

6) Use the formulas on page 1 to compute

$$\int_0^1 \frac{dx}{\sqrt{4x^2+9}} = \int_0^1 \frac{dx}{\sqrt{(2x)^2+(3)^2}} = I$$

A) $\frac{1}{2}\ln(\frac{2}{3})$

B) $\frac{1}{2}\ln\left(\frac{2+\sqrt{13}}{3}\right)$

C) $\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{3}\right)$

D) $\frac{1}{2}\ln\left(\frac{1+\sqrt{3}}{2}\right)$

E) $\frac{1}{2}\ln\left(\frac{2+\sqrt{12}}{2}\right)$

$a=3, \quad u=2x$

thus $du = 2\,dx$, so $dx = \frac{1}{2}du$

$u(0) = 2(0) = 0, \quad u(1) = 2(1) = 2$

$$I = \int_0^2 \frac{\frac{1}{2}du}{\sqrt{u^2+(3)^2}} = \left[\frac{1}{2}\ln\left(u+\sqrt{3^2+u^2}\right)\right]\Big|_0^2$$

$$= \left[\frac{1}{2}\ln(2+\sqrt{13}) - \frac{1}{2}\ln(0+\sqrt{9+0})\right]$$

$$= \frac{1}{2}\ln\left(\frac{2+\sqrt{13}}{3}\right)$$

7) The indefinite integral

$$\int_e^\infty \frac{dx}{x((\ln x)^2+1)} \text{ is equal to}$$

A) $e^2 - 1$

B) $\pi/4$

C) $e \ln 2$

D) $\pi/3$

E) $3e \ln 2$

$$= \lim_{t\to\infty} \int_e^t \frac{dx}{x((\ln x)^2+1)}$$

Let $u = \ln x$. $\to du = \frac{1}{x}\,dx$

$u(e) = \ln e = 1$

$u(t) = \ln t$

$$= \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^2+1} = \lim_{t\to\infty} \left(\tan^{-1} u \Big|_1^{\ln t} \right)$$

$$= \lim_{t\to\infty} \left(\tan^{-1}(\ln t) - \frac{\pi}{4} \right)$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{4}$$

8) Find the length of the arc of the curve $y = \frac{1}{2}x^2$, with $0 \le x \le 1$.

Hint: Use one of the integrals on page 1.

$$\int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+(x)^2}\,dx$$

A) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(2+\sqrt{2})$

B) $\frac{1}{2}\sqrt{2}$

C) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2})$

D) $\sqrt{2} + \ln(2+\sqrt{2})$

E) $\frac{1}{4}\sqrt{2} + \frac{1}{4}\ln(1+\sqrt{2})$

$$\overset{(u=x)}{=} \left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) \right] \Bigg|_0^1$$

$$= \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2}) \right) - (0+0)$$

9) Find the area of the surface obtained by rotating the curve $y = \frac{1}{3}x^3$, $0 \le x \le 1$, about the x-axis.

A) $2\pi(\sqrt{2}-1)$

B) $2\pi(\sqrt{2}-\frac{1}{2})$

C) $2\pi(\sqrt{3}-\frac{2}{3})$

D) $\frac{\pi}{9}(2\sqrt{2}-1)$

E) $\frac{\pi}{6}(2\sqrt{2}-\frac{1}{2})$

$$A = \int_0^1 2\pi\left(\frac{1}{3}x^3\right)\sqrt{1+(x^2)^2}\,dx$$

Let $u = x^4$. Then $du = 4x^3\,dx \to x^3\,dx = \frac{1}{4}\,du$.

$u(0) = 0^4 = 0$. $u(1) = 1^4 = 1$.

$$A = \int_0^1 \frac{2}{3}\pi\sqrt{1+u}\;\frac{1}{4}\,du$$

$$= \frac{2}{3}\pi\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(1+u)^{3/2}\Big|_0^1$$

$$= \frac{1}{9\pi}\left(2^{3/2}-1\right) = \frac{1}{9\pi}\left(2\sqrt{2}-1\right)$$

10) Find the x-coordinate of the centroid of the region of the first quadrant bounded by $y = 1-x^2$, $y = 0$ and the y-axis.

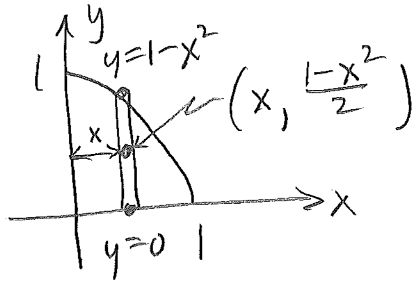
A) 7/12

B) 5/14

C) 5/8

D) 4/9

E) 3/8



$$\bar{x} = \frac{M_y}{m} = \frac{\int_0^1 x(1-x^2)\,dx}{\int_0^1 (1-x^2)\,dx}$$

$$= \frac{\int_0^1 (x-x^3)\,dx}{\int_0^1 (1-x^2)\,dx}$$

$$= \frac{\left(\frac{1}{2}x^2-\frac{1}{4}x^4\right)\Big|_0^1}{\left(x-\frac{1}{3}x^3\right)\Big|_0^1} = \frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{3}} = \frac{\frac{1}{4}}{\frac{2}{3}} = \frac{3}{8}$$

11) The limit of the sequence

$$a_n = \frac{(2n+1)!}{n^2(2n-1)!}$$

is equal to

A) 0

B) 4

C) 3

D) 2

E) The sequence diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n+1)(2n)(2n-1)\cdots(3)(2)(1)}{(n)(n)(2n-1)\cdots(3)(2)(1)}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n)}{(n)(n)}$$

$$= \lim_{n\to\infty} \frac{4n^2+2n}{n^2} = 4$$

12) The sum of the series

$$\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=2}^{\infty} \frac{2^3}{3^2} \cdot \frac{2^{n-2}}{3^{n-2}}$$

is equal to

A) 8/3

B) 7/9

C) 10/3

D) 2/3

E) 4/9

$$= \frac{8}{9} \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^{n-2}$$

$$= \frac{8}{9}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{8}{9}\left(\frac{1}{\frac{1}{3}}\right) = \frac{8}{9}(3) = \frac{8}{3}$$