SOLUTION

MA 161 & 161E **FINAL EXAM** **SPRING 2002**

Name

Student ID Number

Lecturer

Recitation Instructor

Time of Recitation Class

Instructions:

1. The exam has 25 problems, each worth 8 points, for a total of 200 points.
2. Please supply all information requested above.
3. Work only in the space provided, or on the backside of the pages.
4. No books, notes, or calculators are allowed.
5. Use a number 2 pencil on the answer sheet. Print your last name, first name, and fill in the little circles. Under "Section Number", print the division and section number of your recitation class and fill in the little circles. Under "Test/Quiz Number" print 04 and fill in the little circles. Similarly, fill in your student ID and fill in the little circles. Also, fill in your recitation instructor's name; the course, MA 161; and the date, May 1, 2002. Be sure to fill in the circles for each of the answers of the 25 exam questions.
6. Show your work. It may be used if your grade is on the borderline.

1. If $f(x) = \dfrac{\sin x}{x^2}$, then $f'(x) =$

$$f'(x) = \frac{(\cos x)(x^2) - (\sin x)(2x)}{x^4}$$

A. $\dfrac{\cos x}{2x}$

B. $\dfrac{(\cos x)(2x) - (\sin x)(x^2)}{x^4}$

C. $\dfrac{(\cos x)(x^2) - (\sin x)(2x)}{x^4}$

D. $\dfrac{(\sin x)(2x) - (\cos x)(x^2)}{x^4}$

E. $\dfrac{(\sin x)(x^2) - (\cos x)(2x)}{x^4}$

2. If $y = (x^2 + x)\tan x$, then $\dfrac{dy}{dx} =$

$$\frac{dy}{dx} = (2x+1)\tan x + (x^2+1)\sec^2 x$$

A. $(2x+1)\tan x + (x^2+x)\sec^2 x$

B. $(2x+1)\sec^2 x + (x^2+x)\tan x$

C. $(2x+1)\sec^2 x$

D. $(2x+1)\tan x + (x^2+x)\cot x$

E. $(2x+1)\tan x + (x^2+x)\csc^2 x$

3. If $f(x) = \sin(2\pi x^2)$, then $f''\left(\frac{1}{2}\right) =$

$$f'(x) = 4\pi x \cos(2\pi x^2)$$

$$f''(x) = 4\pi \cos(2\pi x^2) + (4\pi x)^2(-\sin(2\pi x^2))$$

$$f''\left(\tfrac{1}{2}\right) = 4\pi \cos\frac{\pi}{2} + 4\pi^2(-\sin\tfrac{\pi}{2})$$

$$= -4\pi^2$$

A. 4π

B. -2π

C. $-2\pi^2$

D. $-4\pi^2$

E. $2\pi^2 + \pi$

4. Which of the following is the derivative of $y = \sqrt{x+1}$ at $a = 2$?

I. $\lim_{x \to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x - 2} = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

II. $\lim_{t \to x} \frac{\sqrt{t+1} - \sqrt{x+1}}{t - x}$

III. $\lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

A. Only I

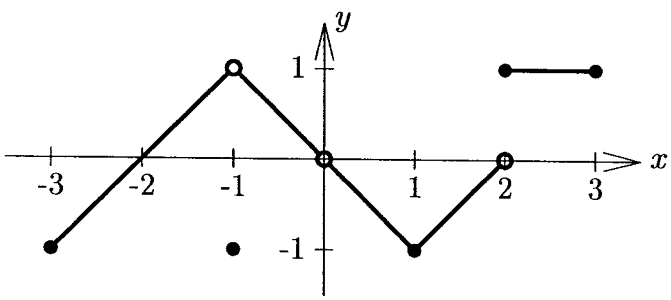
B. Only II

C. Only III

D. Only I and II

E. Only I and III

5. Given the following graph of the function f, which statements are true?



I. $\lim_{x \to -1} f(x) = f(-3)$

II. $\lim_{x \to 0} f(x)$ does not exist

III. $\lim_{x \to 2^-} f(x)$ does not exist

A. Only I

B. Only II

C. Only III

D. Only I and III

E. None are true

$\lim_{x \to -1} f(x) = 1 \neq f(-3)$

$\therefore$ I is false.

$\lim_{x \to 0} f(x) = 0 \quad \therefore$ II is false.

$\lim_{x \to 2^-} f(x) = 0 \quad \therefore$ III is false

6. If $x^2y - xy^3 = 161$ defines y implicitly as a differentiable function of x, then $\dfrac{dy}{dx} =$

$$\frac{d}{dx}(x^2y - xy^3) = 2xy + x^2\frac{dy}{dx} - \left(y^3 + 3xy^2\frac{dy}{dx}\right)$$

$$\frac{d}{dx}(161) = 0$$

$$\therefore \quad 0 = 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx}$$

$$(y^3 - 2xy) = (x^2 - 3xy^2)\frac{dy}{dx}$$

ans is D.

A. 0

B. $2x - 3y^2$

C. $\dfrac{y^3 - x^2}{2x - 3y^2}$

D. $\dfrac{y^3 - 2xy}{x^2 - 3xy^2}$

E. $\dfrac{xy^3 - x^2y}{2x - 3y^2}$

7. Let $f(x) = g(h(x))$ and $h(2) = 3$, $h(3) = 2$, $h'(2) = -1$, $h'(3) = 4$, $g(2) = -2$, $g(3) = -3$, $g'(2) = 5$, and $g'(3) = 6$. Then $f'(2) =$

$$f'(x) = g'(h(x)) \cdot h'(x).$$

$$f'(2) = g'(h(2)) \cdot h'(2)$$

$$= g'(3) \cdot h'(2)$$

$$= 6 \cdot (-1) = -6.$$

A. 15

B. -6

C. 12

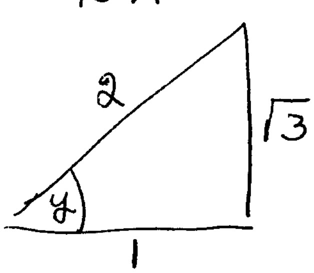
D. -5

E. 6

8. $\tan(\sec^{-1} 2) =$

Let $y = \sec^{-1} 2$.

then $\sec y = 2$



$\therefore \tan y = \sqrt{3}$

A. $\dfrac{\pi}{3}$

B. $\dfrac{\pi}{6}$

C. $\sqrt{3}$

D. $\dfrac{\sqrt{3}}{2}$

E. $\dfrac{1}{2}$

9. The line tangent to the graph of $y = f(x)$ at $(2, 4)$ is parallel to the line tangent to the graph of $y = x^3$ at $x = 1$. Find $f'(2)$.

Let $g(x) = x^3$, $g'(x) = 3x^2$, $g'(1) = 3$.

$f'(2) = g'(1) = 3$

A. 4
B. 3
C. 2
D. 1
E. 0

10. A colony of bacteria, growing at a rate proportional to its population, begins with 150 bacteria. Two hours later the population is 600. How many hours does it take for the population to reach 3000 bacteria?

$P(x) = 150\, e^{kt}$, $P(2) = 600$

$600 = 150\, e^{2k} \to 4 = e^{2k} \to k = \frac{1}{2}\ln 4.$

$3000 = 150\, e^{\ln 4 \cdot \frac{1}{2}t} = 150(4^{t/2}).$

$\to 20 = 4^{t/2} \to \ln 20 = \frac{t}{2}\ln 4$

$\therefore t = \frac{2\ln 20}{\ln 4}.$

A. $\dfrac{2\ln 20}{\ln 4}$
B. $\dfrac{2\ln 4}{\ln 20}$
C. $2\ln 5$
D. $2\ln\left(\dfrac{1}{5}\right)$
E. $\dfrac{2\ln 4}{\ln 2}$

11. $\displaystyle\lim_{x\to -1} \frac{x^2-1}{1-|x|} = \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1}(x-1)$

$x<0 \to |x| = -x.$

$= -2$

A. ∞
B. $-\infty$
C. 2
D. -2
E. 0

12. The number of vertical and horizontal asymptotes of $y = \dfrac{x^2 + \ln|x|}{x^2 - 1}$ is

A. 0

B. 1

C. 2

D. 3

E. 4

$x = \pm 1$ are vertical asymptotes.

$x = 0$ is a vertical asymptote.

$$\lim_{x \to \pm\infty} \frac{x^2 + \ln|x|}{x^2 - 1} \overset{H}{=} \lim_{x \to \pm\infty} \frac{2x + \frac{1}{|x|}}{2x}$$

$$\overset{H}{=} \lim_{x \to \pm\infty} \frac{2 - \frac{1}{x^2}}{2} = 1$$

$\therefore$ $y = 1$ is a horizontal asymptote

13. Let $f(x) = \begin{cases} 3(x-1)^2, & \text{if } x < 0 \\ 2\cos x + a, & \text{if } x \geq 0. \end{cases}$ What value of a makes f continuous at $x = 0$?

A. 2

B. 1

C. 0

D. -1

E. -2

$$\lim_{x \to 0^-} 3(x-1)^2 = 3.$$

$$\lim_{x \to 0^+} (2\cos x + a) = 2 + a.$$

$$\therefore\ 2 + a = 3$$

$$a = 1$$

14. When a stone is dropped in a pool, a circular wave moves out from the point of impact at a rate of 6 in/sec. How fast (in in^2/sec) is the area enclosed by the wave increasing when the radius of the wave is 2 in?

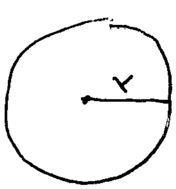
A. 24π

B. 20π

C. 16π

D. 12π

E. None of the above



$A = \pi r^2$ Given $\frac{dr}{dt} = 6$

Find $\frac{dA}{dt}$ when $r = 2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 2 \cdot 6$$

$$= 24\pi \text{ in}^2/\text{sec}.$$

15. Find the difference between the local maximum and the local minimum values of the function $f(x) = x^3 - 3x + 27$.

$f'(x) = 3x^2 - 3 = 0 \to x = \pm 1.$

$f''(x) = 6x \quad \therefore$ Local Max when $x = -1$

Local Min when $x = 1$.

$f(-1) - f(1) = 29 - 25 = 4$

A. 6

B. 4

C. 2

D. 1

E. None of the above

16. The absolute maximum of $f(x) = (1-x)\sqrt{x}$ on the interval $[0, 4]$ is

$f'(x) = -\sqrt{x} + (1-x)\frac{1}{2\sqrt{x}} = 0$

$= -2x + 1 - x = 0 \to x = \frac{1}{3}$

$x = 0 \qquad x = \frac{1}{3} \qquad x = 4$

$f(0) = 0 \qquad f(\frac{1}{3}) = \frac{2}{3} \cdot \frac{1}{\sqrt{3}} \qquad f(4) = -6$

Max.

A. 0

B. $\frac{2}{\sqrt{3}}$

C. $\frac{\sqrt{3}}{9}$

D. $\frac{1}{3}$

E. $\frac{2}{3\sqrt{3}}$

17. If $-\frac{\pi}{2} \le x \le \pi$, then the largest interval on which $f(x) = x\sin x + \cos x$ is increasing is

$f'(x) = \sin x + x\cos x - \sin x = x\cos x.$

$-\frac{\pi}{2} < x < 0. \qquad 0 < x < \frac{\pi}{2} \qquad \frac{\pi}{2} < x < \pi$

$f'(x) < 0 \qquad f'(x) > 0 \qquad f'(x) < 0$

f is increasing.

A. $\left(-\frac{\pi}{2}, 0\right)$

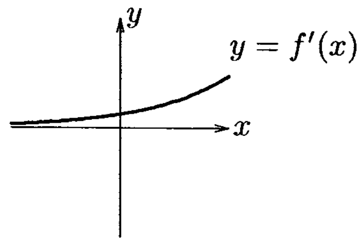
B. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

C. $\left(0, \frac{\pi}{2}\right)$

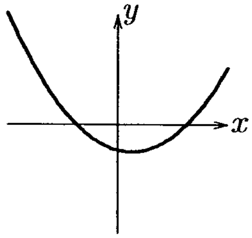
D. $\left(\frac{\pi}{2}, \pi\right)$

E. $(0, \pi)$

18. Given the graph $y = f'(x)$ below, select a graph which best represents the graph of $y = f(x)$.



A.



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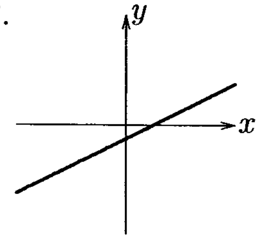
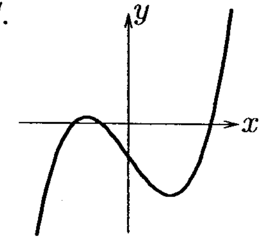
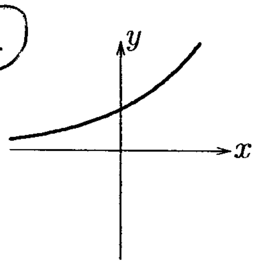


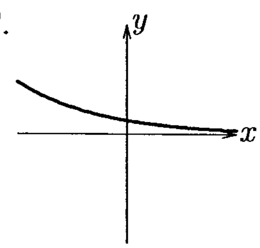
C.



D.



E.

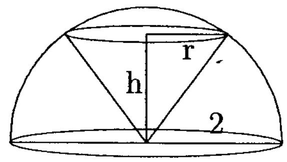


$f'(x) > 0$ $\therefore$ f is increasing. so only B or D are possible.

$\lim_{x \to -\infty} f'(x) = 0$ $\therefore$ $\lim_{x \to -\infty} f(x) = 0$.

$\therefore$ the ans is D

19. A right circular cone is inscribed in a hemisphere of radius 2 as shown below.



Find the ratio of height to radius, $\frac{h}{r}$, of the cone that has the maximum volume. $\left(V = \frac{1}{3}\pi r^2 h\right)$.

Maximize $V = \frac{1}{3}\pi r^2 h$.
$h^2 + r^2 = 4$.

$\therefore V = \frac{1}{3}(4-h^2)\cdot h = \frac{1}{3}(4h - h^3)$.

$\frac{dV}{dh} = \frac{1}{3}(4 - 3h^2) = 0$, $\therefore h^2 = \frac{4}{3}$

But $h > 0$ so $h = \frac{2}{\sqrt{3}}$. $\left(\frac{d^2V}{dh^2} < 0 \text{ if } h = \frac{2}{\sqrt{3}}\right)$

$\therefore r^2 = 4 - h^2 = 4 - \frac{4}{3} = \frac{8}{3}$.

$r = \frac{2\sqrt{2}}{\sqrt{3}}$ $\therefore \frac{h}{r} = \frac{\frac{2}{\sqrt{3}}}{\frac{2\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}$.

A. $\frac{1}{2}$

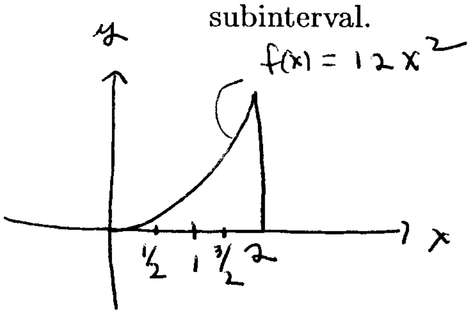
B. 2

C. $\frac{1}{\sqrt{2}}$

D. $\sqrt{2}$

E. $\frac{2}{3}$

20. Let $f(x) = 12x^2$ on $[0, 2]$. Let the interval be divided into 2 equal subintervals. Find the Riemann sum for this partition, $\sum_{i=1}^{n} f(x_i^*)\Delta x$, where x_i^* is the midpoint of its subinterval.



$\Delta x = 1$.
$f\left(\frac{1}{2}\right) = 3$.
$f\left(\frac{3}{2}\right) = 27$.

A. 60

B. 30

C. 8

D. 15

E. 45

$\therefore f\left(\frac{1}{2}\right)\Delta x + f\left(\frac{3}{2}\right)\Delta x = 30$

21. If $\int_0^1 f(t)dt = 7$ and $\int_0^3 f(t)dt = 6$, then $\int_1^3 f(t)dt =$

$\int_0^3 f(t)dt = \int_0^1 f(t)dt + \int_1^3 f(t)dt$

$6 = 7 + \int_1^3 f(t)dt$

$\therefore \int_1^3 f(t)dt = -1$

A. -1

B. 1

C. 13

D. 3

E. Cannot be determined

22. If $f''(x) = x + \sqrt{x}$, $f(0) = 1$, and $f'(0) = 2$, find $f(1)$.

$f'(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + C_1$

$2 = C_1$

$\therefore f'(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + 2$

$f(x) = \frac{1}{6}x^3 + \frac{4}{15}x^{5/2} + 2x + C_2$.

$1 = C_2 \quad \therefore f(x) = \frac{1}{6}x^3 + \frac{4}{15}x^{5/2} + 2x + 1$.

$\therefore f(1) = \frac{1}{6} + \frac{4}{15} + 2 + 1 = 3 + \frac{13}{30}$

A. $\frac{13}{30}$

B. $3\frac{1}{6}$

C. $1\frac{1}{6}$

D. $4\frac{1}{6}$

E. $3\frac{13}{30}$

23. If $F(x) = \int_0^{\sqrt{x}} \cos(t^2)dt$ then $F'(x) =$

$F'(x) = \cos x \cdot \frac{1}{2\sqrt{x}}$

A. $-\sin x$

B. $\sqrt{x}\cos x$

C. $\frac{\cos x}{2\sqrt{x}}$

D. $-\sqrt{x}\sin x^2$

E. $\frac{\cos x^2}{\sqrt{x}}$

24. $\int_0^{\frac{\pi}{4}} \sin^2 x \cos x dx =$ $\int_0^{\sqrt{2}/2} u^2\, du = \frac{1}{3}u^3\Big]_0^{\sqrt{2}/2}$

Let $u = \sin x$, $du = \cos x\, dx$

$x = 0$, $u = 0$

$x = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$

$= \frac{2\sqrt{2}}{3 \cdot 8} = \frac{\sqrt{2}}{12}$

A. $\frac{\sqrt{2}}{12}$

B. $\frac{\pi}{2}$

C. $\frac{\sqrt{2}}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi\sqrt{2}}{6}$

25. $\int_{-1}^{1} \frac{x}{\sqrt{x^2+1}} dx =$ I

A. $2(\sqrt{2}-1)$

B. $\sqrt{2}-1$

C. 2

D. $1-\sqrt{2}$

E. 0

$f(x) = \frac{x}{\sqrt{x^2+1}}$ is an odd function.

$f(-x) = -f(x)$

$\therefore \int_{-1}^{1} \frac{x}{\sqrt{x^2+1}} dx = 0.$

Alternatively, Let $u = x^2+1$ then $du = 2x\,dx$

so $x\,dx = \frac{1}{2}du$.

If $x = -1$, $u = 2$

$x = 1$, $u = 2$

$\therefore I = \frac{1}{2}\int_2^2 \frac{du}{\sqrt{u}} = 0$