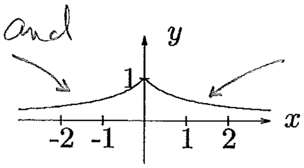
SOLUTIONS

MA 16100 **Exam II** **Fall 2008**

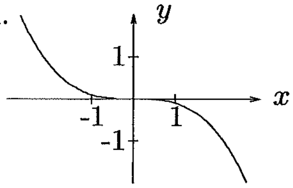
1. The graph of $y = f(x)$ is shown below.

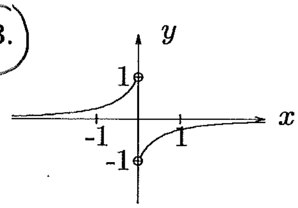
Slope positive and approaching 1 as $x \to 0^-$



slope negative and approaching -1 as $x \to 0^+$

Which of the following could be the graph of $y = f'(x)$?

A.


B.


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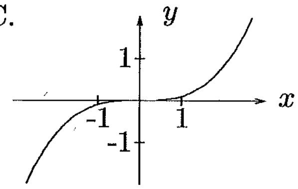

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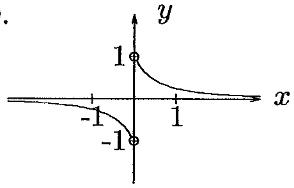
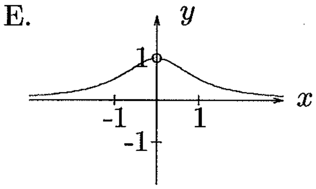


E.


2. Let $f(x) = \begin{cases} x, & x \le -1 \\ x+1, & -1 < x \le 0 \\ x^2+1, & 0 < x \le 1 \\ 2x, & 1 < x \end{cases}$

Find the points at which f is NOT differentiable.

A. $x = 0$, $x = 1$, and $x = -1$

B. $x = 0$ and $x = 1$

C. $x = 0$ and $x = -1$

D. $x = 1$ and $x = -1$

E. $x = 0$

f differentiable at all x except possibly -1, 0 and 1,

f not differentiable at $x=-1$ since f not continuous at $x=-1$.

$\lim_{x\to -1^-} f(x) = -1$, but $\lim_{x \to -1^+} f(x) = 0$

f not differentiable at $x=0$ since

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{(x+1)-(1)}{x-0} = 1$

but $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{(x^2+1)-(1)}{x-0} = 0$

1

f is differentiable at $x=1$ since

$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{(x^2+1)-(2)}{x-1} = 2$

and

$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(2x)-(2)}{x-1} = 2$

3. At time 0 a ball is thrown directly upward from a platform 10m tall. Its height above the ground after t seconds is $s = -5t^2 + 5t + 10$, where s is in meters. The ball hits the ground after 2 seconds. What is its velocity at impact?

$V(t) = \frac{ds}{dt} = -10t + 5$

$V(2) = -10(2) + 5 = -15$

A. 0

B. -5 m/s

C. -10 m/s

D. -15 m/s

E. -20 m/s

4. At which point(s) does the curve $y = x^3 - 6x^2 + 12x + 7$ have a horizontal tangent?

Solve $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 3x^2 - 12x + 12$

$3(x^2 - 4x + 4) = 0$

$\rightarrow 3(x-2)(x-2) = 0$

$\rightarrow x = 2$

A. $x = 0$ and $x = 1$

B. $x = 1$ and $x = 2$

C. $x = 0$ and $x = 2$

D. $x = 1$

E. $x = 2$

5. If $f(x) = \sqrt{x}\, e^{x-4}$, then $f'(4) =$

A. $\frac{9}{4}$

B. $\frac{1}{2}$

C. 0

D. $\frac{5}{4}$

E. $\frac{3}{4}$

$$f'(x) = \frac{1}{2\sqrt{x}} e^{x-4} + \sqrt{x}\, e^{x-4}$$

$$f'(4) = \frac{1}{2\sqrt{4}} e^{4-4} + \sqrt{4}\, e^{4-4}$$

$$= \frac{1}{4} e^{0} + 2e^{0}$$

$$= \frac{1}{4} + 2$$

$$= \frac{9}{4}$$

6. If $f(x) = (1 + \sin 2x)^{10}$, then $f'\left(\frac{\pi}{2}\right) =$

A. 1

B. 10

C. -10

D. 20

E. -20

$$f'(x) = 10(1+\sin 2x)^{9}(0+2\cos 2x)$$

$$f'\left(\frac{\pi}{2}\right) = 10(1+\sin \pi)^{9}(2\cos \pi)$$

$$= 10(1+0)^{9}(2(-1))$$

$$= -20$$

7. If $g(x) = \tan\left(\frac{\pi}{2}\, f(x)\right)$, where $f(0) = 0$ and $f'(0) = 2$, then $g'(0) =$

$$g'(x) = \left[\sec^2\left(\frac{\pi}{2} f(x)\right)\right]\left[\frac{\pi}{2}\cdot f'(x)\right]$$

$$g'(0) = \left[\sec^2\left(\ 0\ \right)\right]\left[\frac{\pi}{2}\cdot 2\right]$$

$$= \left[\ 1\ \right]\left[\ \pi\ \right]$$

$$= \pi$$

A. 4

B. $\frac{\pi}{2}$

C. π

D. 2

E. Cannot be determined

8. If $f(x) = \ln\sqrt{\frac{x^3}{1-x^2}}$, then $f'(x) =$

$$f(x) = \ln\sqrt{\frac{x^3}{1-x^2}} = \ln\left(\frac{x^3}{1-x^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln\left(\frac{x^3}{1-x^2}\right) = \frac{1}{2}\left(\ln(x^3) - \ln(1-x^2)\right)$$

$$\rightarrow f'(x) = \frac{1}{2}\left(\frac{3x^2}{x^3} - \frac{-2x}{1-x^2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{x} + \frac{2x}{1-x^2}\right)$$

A. $\frac{1}{2}\left(\frac{3}{x} - 1\right)$

B. $\frac{1}{2}\left(\frac{3}{x} + \frac{2x}{1-x^2}\right)$

C. $\frac{1}{2}\left(\frac{3}{x} - \frac{2x}{1-x^2}\right)$

D. $\frac{1}{2}\left(\frac{3}{x} + 1\right)$

E. $\frac{1}{2}\left(\frac{5}{x}\right)$

9. Find an equation for the line tangent to the graph of $y = \frac{x^3}{\ln x}$ at the point (e, e^3).

A. $y = 2e^2x - e^3$

B. $y = 2e^2x + e^3 - 3$

C. $y = 2e^2x + e$

D. $y = -e^2x + e$

E. $y = -e^2x - e$

$$\frac{dy}{dx} = \frac{3x^2 \ln x - x^3\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=e} = \frac{3e^2 \ln e - e^2}{(\ln e)^2} = \frac{2e^2}{1} = 2e^2$$

tangent line: $y - e^3 = 2e^2(x - e)$

$\rightarrow y = 2e^2x - 2e^3 + e^3 = 2e^2x - e^3$

10. Use implicit differentiation to find $\frac{dy}{dx}$ at the point $(1, 2)$ if $x^4 - 3x^2y + y^2 + y^3 = 7$.

A. $\frac{-2}{5}$

B. $\frac{1}{2}$

C. $\frac{4}{13}$

D. $-\frac{3}{5}$

E. $\frac{8}{13}$

$\frac{d}{dx} \rightarrow 4x^3 - \left(6xy + 3x^2\frac{dy}{dx}\right) + 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$(1,2) \rightarrow 4 - \left(12 + 3\frac{dy}{dx}\right) + 4\frac{dy}{dx} + 12\frac{dy}{dx} = 0$

$\rightarrow 13\frac{dy}{dx} = 8$

$\rightarrow \frac{dy}{dx} = \frac{8}{13}$

11. Let $y = x^{\tan x}$. Find $\frac{dy}{dx}$.

A. $\frac{dy}{dx} = x^{\tan x}(\sec x \tan x \ln x + \tan x)$

B. $\frac{dy}{dx} = x^{\tan x}(\sec^2 x)\left(\frac{1}{x}\right)$

C. $\frac{dy}{dx} = x^{\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$

D. $\frac{dy}{dx} = x^{\tan x - 1}(\tan x)$

E. $\frac{dy}{dx} = x^{\tan x}(\sec^2 x)$

$\rightarrow \ln y = \ln\left(x^{\tan x}\right) = (\tan x)(\ln x)$

$\rightarrow \frac{\frac{dy}{dx}}{y} = (\sec^2 x)(\ln x) + (\tan x)\left(\frac{1}{x}\right)$

$\rightarrow \frac{dy}{dx} = x^{\tan x}\left((\sec^2 x)(\ln x) + \frac{\tan x}{x}\right)$

12. A spherical balloon increases in radius by $\frac{1}{2}$ inch per minute. Find the average rate of change of the volume of the balloon $\left(\frac{\text{inches}^3}{\text{minute}}\right)$ when the radius increases from 2 inches to 4 inches (Volume of sphere: $V = \frac{4}{3}\pi r^3$).

A. $28\pi \, \frac{\text{inches}^3}{\text{minute}}$

B. $48\pi \, \frac{\text{inches}^3}{\text{minute}}$

C. $16\pi \, \frac{\text{inches}^3}{\text{minute}}$

D. $\frac{64\pi}{3} \, \frac{\text{inches}^3}{\text{minute}}$

E. $\frac{56\pi}{3} \, \frac{\text{inches}^3}{\text{minute}}$

$$\frac{\Delta V}{\Delta t} = \frac{V(4) - V(2)}{4-2} \cdot \frac{1}{2}$$

$$= \frac{\frac{4}{3}\pi 4^3 - \frac{4}{3}\pi 2^3}{2} \cdot \frac{1}{2}$$

$$= \frac{\frac{4}{3}\pi}{2}(64 - 8) \cdot \frac{1}{2}$$

$$= \frac{2}{3}\pi(56) \cdot \frac{1}{2}$$

$$= \frac{56\pi}{3}$$

note: $\frac{\Delta V}{\Delta t} = \frac{\Delta V}{\Delta r} \cdot \frac{\Delta r}{\Delta t}$

and $\frac{\Delta r}{\Delta t} = \frac{1}{2}$

13. 60% of a radioactive substance decays in 3 hours. What is the half–life of the substance?

A. $3\left(\ln \frac{1}{5}\right)$ hours

B. $3\left(\dfrac{\ln \frac{1}{2}}{\ln \frac{2}{5}}\right)$ hours

C. $3\left(\dfrac{\ln \frac{1}{2}}{\ln \frac{5}{2}}\right)$ hours

D. $3\left(\dfrac{\ln \frac{5}{2}}{\ln \frac{1}{2}}\right)$ hours

E. $3\left(\dfrac{\ln \frac{2}{5}}{\ln \frac{1}{2}}\right)$ hours

$A(t) = A(0)\, e^{kt}$

$\rightarrow A(3) = 0.4\, \cancel{A(0)} = \cancel{A(0)}\, e^{k3}$

$\rightarrow \ln 0.4 = 3k \rightarrow k = \frac{\ln 0.4}{3}$

$\rightarrow A(t) = A(0)\, e^{\frac{\ln 0.4}{3} t}$

$\frac{1}{2} \cancel{A(0)} = \cancel{A(0)}\, e^{\frac{\ln 0.4}{3} t_{half-life}}$

$\rightarrow \ln \frac{1}{2} = \frac{\ln 0.4}{3} t_{half-life}$

$\rightarrow t_{half-life} = 3\left(\frac{\ln \frac{1}{2}}{\ln 0.4}\right) = 3\left(\frac{\ln \frac{1}{2}}{\ln \frac{2}{5}}\right)$

14. Two sides of a triangle are 3 in. and 7 in. and the angle between them is increasing at 0.2 radians per minute. Find the rate at which the area of the triangle is increasing when the angle between the sides is $\frac{\pi}{6}$?

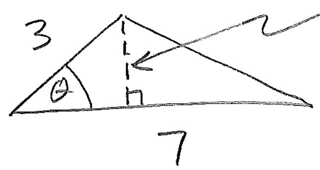
A. $\frac{21\sqrt{3}}{5}\ \frac{\text{inches}^2}{\text{minute}}$

B. $\frac{21}{10}\ \frac{\text{inches}^2}{\text{minute}}$

C. $\frac{21\sqrt{3}}{10}\ \frac{\text{inches}^2}{\text{minute}}$

D. $\frac{21}{20}\ \frac{\text{inches}^2}{\text{minute}}$

E. $\frac{21\sqrt{3}}{20}\ \frac{\text{inches}^2}{\text{minute}}$



$3 \sin \theta$

Know: $\frac{d\theta}{dt} = \frac{1}{5}\ \frac{\text{radian}}{\text{min}}$

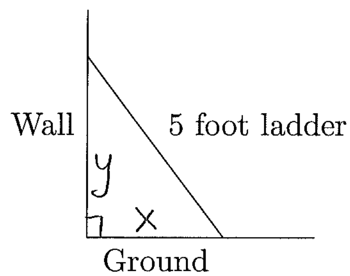
want: $\frac{dA}{dt}$ when $\theta = \frac{\pi}{6}$

$A = \frac{1}{2}(7)(3 \sin \theta)$

$\rightarrow \frac{dA}{dt} = \frac{21}{2}(\cos \theta)\left(\frac{d\theta}{dt}\right)$

$\theta = \frac{\pi}{6},\ \frac{d\theta}{dt} = \frac{1}{5} \rightarrow \frac{dA}{dt} = \left(\frac{21}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{5}\right) = \frac{21\sqrt{3}}{20}$

15. A 5 foot ladder standing on level ground leans against a vertical wall. The bottom of the ladder is pulled away from the wall at 2 ft/sec. How fast is the AREA under the ladder changing when the top of the ladder is 4 feet above the ground?



Know: $\frac{dx}{dt} = 2 \frac{ft}{sec}$

Want: $\frac{dA}{dt}$ when $y=4$

A. 25 ft^2/sec

B. $\frac{7}{4}$ ft^2/sec

C. -6 ft^2/sec

D. $-\frac{3}{2}$ ft^2/sec

E. $-\frac{25}{4}$ ft^2/sec

$$A = \frac{1}{2}xy \longrightarrow \frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right)$$

$x^2+y^2=25$. Need x and $\frac{dy}{dt}$ when $y=4$

$$y=4 \longrightarrow x = \sqrt{25-4^2} = 3$$

$$y=\sqrt{25-x^2} \longrightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{25-x^2}}(-2x)\left(\frac{dx}{dt}\right) = \frac{-\cancel{2}x}{\cancel{2}y}\frac{dx}{dt}$$

$$x=3,\ y=4,\ \frac{dx}{dt}=2 \rightarrow \frac{dy}{dt} = \frac{-3}{4}(2) = -\frac{3}{2}$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right) = \frac{1}{2}\left((2)(4)+(3)\left(-\frac{3}{2}\right)\right)$$

$$= \frac{1}{2}\left(8-\frac{9}{2}\right) \underset{8}{=} \frac{1}{2}\left(\frac{16-9}{2}\right) = \frac{7}{4}$$