MA26500 FINAL EXAM Fall 2023

GREEN - Test Version 01

NAME ______________________ INSTRUCTOR ______________________

1. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Use the back of the test pages for scratch paper. **PLEASE PUT YOUR SCANTRON UNDERNEATH YOUR QUESTION SHEETS WHEN YOU ARE NOT FILLING IN THE SCANTRON SHEET.**

2. Fill in your name and your instructor's name on the question sheets (above).

3. You must use a **#2 pencil** on the mark–sense sheet (answer sheet). Fill in the instructor's name and the course number(**MA265**), fill in the correct TEST/QUIZ NUMBER (**GREEN is 01**), your name, your section number (see below if you are not sure), and your 10-digit PUID(BE SURE TO INCLUDE THE TWO LEADING ZEROS of your PUID.) and shade them in the appropriate spaces. Sign the mark–sense sheet.

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|---|---|---|---|---|---|---|---|
| 101 | MWF | 12:30PM | Ying Zhang | 600 | MWF | 10:30AM | Farrah Yhee |
| 102 | MWF | 11:30AM | Ying Zhang | 601 | MWF | 11:30AM | Farrah Yhee |
| 153 | MWF | 1:30PM | Ying Zhang | 650 | MWF | 11:30AM | Yiran Wang |
| 154 | MWF | 11:30AM | Takumi Murayama | 651 | MWF | 12:30PM | Yiran Wang |
| 205 | MWF | 11:30AM | Jing Wang | 701 | TR | 12:00PM | Raechel Polak |
| 206 | MWF | 2:30PM | Sam Nariman | 702 | TR | 4:30PM | Vaibhav Pandey |
| 357 | MWF | 8:30AM | Jonathon Peterson | 703 | MWF | 10:30AM | Takumi Murayama |
| 410 | MWF | 11:30AM | Siamak Yassemi | 704 | MWF | 8:30AM | Anurag Sahay |
| 451 | MWF | 1:30PM | Siamak Yassemi | 705 | MWF | 1:30PM | Yi Wang |
| 501 | MWF | 1:30PM | Yilong Zhang | 706 | MWF | 12:30PM | Yi Wang |
| 502 | MWF | 10:30AM | Yilong Zhang | 707 | TR | 3:00PM | Vaibhav Pandey |

4. There are 25 questions, each is worth 8 points. Show your work on the question sheets. also **CIRCLE** your answer choice for each problem on the question sheets in case your scantron is lost. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

5. **Please remain seated during the last 10 minutes of the exam.** When time is called, all students must put down their writing instruments immediately.

6. **Turn in both the mark-sense sheets and the question sheets to your instructor when you are finished**.

**1.** Let $A$ be an $5 \times 7$ matrix. Which of the following statements must be **FALSE**?

A. The columns of $A$ are linearly dependent.

B. If $\text{rank}(A) = 5$, then every equation of the form $A\mathbf{x} = \mathbf{b}$ has a unique solution.

C. The row space of $A$ can have any dimension between 0 and 5.

D. $\dim(\text{Row}(A)) + \text{nullity}(A^T) = 5$.

E. $\text{Col}(A)$ is a subspace of $\mathbb{R}^5$.

**2.** For which of the following $3 \times 3$ matrices $A$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for *every* $\mathbf{b}$ in $\mathbb{R}^3$?

A. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

B. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

C. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

D. $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 4 \end{bmatrix}$

E. $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{bmatrix}$

**3.** Which of the following sets of vectors is a basis for the nullspace of the following matrix?

$$\begin{bmatrix} 0 & 3 & 15 & 10 & 27 & -4 \\ 0 & 2 & 10 & 7 & 19 & -2 \\ 0 & 1 & 5 & 3 & 8 & -1 \end{bmatrix}$$

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}, \begin{bmatrix} 27 \\ 19 \\ 8 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}, \begin{bmatrix} 27 \\ 19 \\ 8 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

**4.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation for which the image of a vector $\mathbf{x} \in \mathbb{R}^2$ is defined by reflecting $\mathbf{x}$ across the line $x_1 = -x_2$ and then multiplying the first coordinate by 3 to obtain $T(\mathbf{x})$. Let $A$ be the matrix transformation for $T$, find matrix $A$.

A. $\begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$

**5.** Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ be a linear transformation and $A$ be the matrix transformation for $T$. Which of the following statements must be TRUE?

A. $A$ can have rank 5.

B. The kernel of $T$ is a subspace of $\mathbb{R}^4$.

C. $T$ is onto.

D. $T$ is not one-to-one.

E. The vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in the range of $T$.

**6.** What is the area of the parallelogram with vertices at $(0,-1)$, $(-1,2)$, $(5,0)$, and $(4,3)$.

A. 8

B. 14

C. 16

D. $2\sqrt{65}$

E. 20

**7.** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T(\begin{bmatrix}2\\1\end{bmatrix}) = \begin{bmatrix}4\\5\\6\end{bmatrix}, \; T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

What is $T(\begin{bmatrix}0\\1\end{bmatrix})$?

A. $\begin{bmatrix}5\\7\\9\end{bmatrix}$

B. $\begin{bmatrix}-3\\-3\\-3\end{bmatrix}$

C. $\begin{bmatrix}-\frac{2}{3}\\-\frac{1}{3}\\0\end{bmatrix}$

D. $\begin{bmatrix}\frac{1}{2}\\1\\\frac{3}{2}\end{bmatrix}$

E. $\begin{bmatrix}7\\8\\9\end{bmatrix}$

**8.** Compute the determinant of the following matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & -2 \\ -1 & -1 & 3 & -2 \\ -3 & 1 & -5 & -1 \\ 0 & 0 & 4 & 0 \end{bmatrix}.$$

A. 0

B. 4

C. 16

D. $-32$

E. 32

**9.** Consider the matrix

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Let $B$ be the inverse of the matrix $A$, and $B = [b_{ij}]$. Compute $b_{32}$.

A. $\frac{1}{6}$

B. $-\frac{1}{4}$

C. $\frac{1}{3}$

D. $\frac{1}{12}$

E. $-\frac{1}{2}$

**10.** Given a matrix $A = \begin{bmatrix} 8 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}$. Which of the following sets describe a subspace in $\mathbb{R}^3$?

(i). The set of all vectors $\mathbf{x}$ that solves the linear system $A\mathbf{x} = \mathbf{0}$

(ii). The set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 5\mathbf{x}$ holds.

(iii). The eigenspace correponding to the eigenvalue 8.

(iv). The set of all vectors that are of the form $\left\{ \begin{bmatrix} s+t \\ t \\ 1+s \end{bmatrix}, s, t \in \mathbb{R}. \right\}$

(v). The set of all vectors that are of the form $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x \leq 0, y \leq 0 \right\}$.

A. (i) and (iii) only

B. (i), (ii), and (iii) only

C. (i), (ii), (iii), and (v) only

D. (ii), (iv), and (v) only

E. (ii), (iii),and (v) only

**11.** Let

$$A = \begin{bmatrix} 1 & 2 & a-1 \\ a & 1 & 0 \\ 1 & a & 0 \end{bmatrix}.$$

Find all the values of $a$ such that $A\mathbf{x} = \mathbf{0}$ has some nontrivial solutions.

A. 1 only

B. $-1$ only

C. 0 or $-1$

D. 1 or $-1$

E. 0 or 1

**12.** Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 3 \end{bmatrix}.$$

Which of the following is a basis for range $T$?

A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

**13.** Let $V$ be the subspace of polynomials $p(t)$ in $\mathbb{P}_3$ satisfying $p(0) = 0$ and $p'(1) = 0$. What is the dimension of $V$?

A. 1

B. 2

C. 3

D. 4

E. 0

**14.** Which of the following matrices is/are diagonalizable over real numbers?

(i) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -144 & 10 \\ 3 & 10 & 144 \end{bmatrix}$

A. (ii) only

B. (i) and (ii) only

C. (iii) and (iv) only

D. (i), (iii) and (iv) only

E. (i) and (iii) only

**15.** Suppose that $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are eigenvectors of matrix $A = \begin{bmatrix} -1 & a & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & b \\ 1 & -1 & 4 & -4 \end{bmatrix}$. Find $a$ and $b$.

A. $a = -1, b = 0$

B. $a = 1, b = 0$

C. $a = 1, b = -1$

D. $a = 0, b = 1$

E. $a = 0, b = -1$

**16.** Let $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$. Then $\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}$ is

A. an eigenvector of $A$ corresponding to eigenvalue $-2-3i$.

B. an eigenvector of $A$ corresponding to eigenvalue $2-3i$.

C. an eigenvector of $A$ corresponding to eigenvalue $2+3i$.

D. an eigenvector of $A$ corresponding to eigenvalue $-2+3i$.

E. an eigenvector of $A$ corresponding to eigenvalue 3i.

**17.** Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. Note that the zero polynomial is in $\mathbb{P}_2$. Let $\mathcal{B} = \{1-t, 1+t, t+t^2\}$ be an ordered basis for $\mathbb{P}_2$ and $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be a linear map such that the matrix of $T$ relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}.$$

If $p(t)$ is a polynomial in $\mathbb{P}_2$ such that the coordinate of $p(t)$ relative to the basis $\mathcal{B}$ is $[\mathrm{p(t)}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, find $T(\mathrm{p(t)})$.

A. $t^2 + 3t + 4$

B. $t + 2$

C. $3t^2 + 6t + 1$

D. $t^2 + 3t + 1$

E. 5

**18.** For the system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$, the origin is

A. a repeller

B. an attractor

C. a saddle point

D. a spiral point

E. none of the above

**19.** Given that vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$.
Let $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a particular solution to the initial value problem

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Find $x(3) + y(3)$.

A. $5e^3 + 2e^3$

B. $2e^3 + 2e^6$

C. $3e + 4e^2$

D. $5e^3 + 4e^3$

E. $5e^3 + 4e^6$

**20.** Let $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Then the distance from $\mathbf{y}$ to $W$ is

A. $2\sqrt{3}$

B. $3\sqrt{3}$

C. $3\sqrt{2}$

D. $\sqrt{6}$

E. $2\sqrt{6}$

**21.** Let $a, b, c$ be real numbers and $W$ be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} a \\ b \\ c-1 \end{bmatrix}, \begin{bmatrix} -a \\ b-1 \\ c \end{bmatrix}$.

If $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is in $W^{\perp}$, then $a =$

A. $0$

B. $\frac{1}{2}$

C. $\frac{3}{2}$

D. $-\frac{1}{2}$

E. $-\frac{3}{2}$

**22.** Suppose $A$ is an $n \times n$ matrix. Which of the following statements is FALSE?

A. $A$ is a nonsingular matrix if and only if $A^T$ is a nonsingular matrix.

B. $A$ is an orthogonal matrix if and only if $A^T$ is an orthogonal matrix.

C. $A$ is a symmetric matrix if and only if $A^T$ is a symmetric matrix.

D. $A$ is orthogonally diagonalizable if and only if $A^T$ is orthogonally diagonalizable.

E. $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $A^TA\mathbf{x} = A^T\mathbf{b}$ is consistent.

**23.** Performing the Gram-Schmidt process on the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ yields an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. What is $\mathbf{u}_3$?

A. $\frac{1}{\sqrt{110}} \begin{bmatrix} 4 \\ -3 \\ 7 \\ -6 \end{bmatrix}$

B. $\frac{1}{\sqrt{110}} \begin{bmatrix} 3 \\ 4 \\ 7 \\ 6 \end{bmatrix}$

C. $\frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

D. $\frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$

E. $\frac{1}{\sqrt{110}} \begin{bmatrix} -6 \\ 7 \\ -3 \\ 4 \end{bmatrix}$

**24.** Find a least-squares solution of an inconsistent system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$.

A. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$

C. $\begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}$

D. $\begin{bmatrix} 3 \\ -\frac{1}{2} \end{bmatrix}$

E. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

**25.** Let $C[-1, 1]$ be the space of all continuous functions over $[-1, 1]$ with the inner product

$$< f(t), g(t) >= \int_{-1}^{1} f(t)g(t)dt$$

for any $f(t), g(t) \in C[-1, 1]$. Find values $a$ and $b$ so that $\{1, t+a, t^2+b\}$ is an orthogonal set.

A. $a = 1, b = \frac{2}{3}$

B. $a = 1, b = -\frac{1}{3}$

C. $a = 0, b = \frac{1}{3}$

D. $a = 0, b = -\frac{1}{3}$

E. $a = 0, b = -\frac{2}{3}$

Scratch paper

Scratch paper