MA 16500
FINAL EXAM VERSION 01
December 13, 2023

INSTRUCTIONS

Your name ____________________ Your TA's name ____________________

Student ID # ________________ Section # and recitation time __________

1. You must use a #2 pencil on the scantron sheet (answer sheet).

2. Check that the cover of your exam booklet is GREEN and that it has VERSION 01 on the top. Write 01 in the TEST/QUIZ NUMBER boxes and blacken in the appropriate spaces below.

3. On the scantron sheet, fill in your **TA's name, i.e., the name of your recitation instructor (NOT the lecturer's name)** and the course number.

4. Fill in your **NAME** and **PURDUE ID NUMBER**, and blacken in the appropriate spaces. Put 00 at the front of PUID to make it a 10 digit number, and then fill it in.

5. Fill in the four-digit **SECTION NUMBER**. Your section number is a 3 digit number. Put 0 at the front to make it a 4 digit number, and then fill it in.

6. Sign the scantron sheet.

7. Blacken your choice of the correct answer in the space provided for each of the questions 1–25. While mark all your answers on the scantron sheet, you should **show your work** on the exam booklet. Although no partial credit will be given, any disputes about the grade or grading will be settled by examining your written work on the exam booklet.

8. There are 25 questions, each of which is worth 8 points. The maximum possible score is: 25 questions × 8 points = 200 points.

9. NO calculators, electronic device, books, or papers are allowed. Use the back of the test pages for scrap paper.

10. After you finish the exam, **turn in BOTH the scantron sheet and the exam booklet**.

11. If you finish the exam before 5:25 PM, you may leave the room after turning in the scantron sheet and the exam booklet. **If you don't finish before 5:25 PM, you should REMAIN SEATED** until your TA comes and collects your scantron sheet and exam booklet.

Exam Policies

1. There is no individual seating. Just follow TAs' seating instructions.
2. Students may not open the exam until instructed to do so.
3. No student may leave in the first 20 min or in the last 5 min of the exam.
4. Students late for more than 20 min will not be allowed to take the exam; they will have to contact their lecturer within one day for permission to take a make-up exam.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs/proctors will collect the scantron sheet and the exam booklet.
6. Any violation of the above rules may result in score of zero.

Rules Regarding Academic Dishonesty

1. You are not allowed to seek or obtain any kind of help from anyone to answer questions on the exam. If you have questions, consult only your instructor.
2. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you have finished your exam, handed it in to your instructor and left the room.
3. You may not consult notes, books, calculators. You may not handle cell phones or cameras, or any electronic devices until after you have finished your exam, handed it in to your instructor/proctor and left the room.
4. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported immediately to the Office of the Dean of Students.

I have read and understand the exam policies and the rules regarding the academic dishonesty stated above:

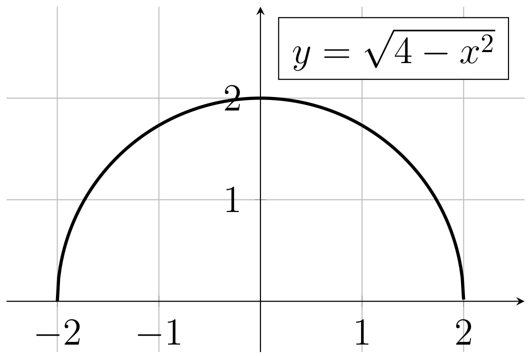
STUDENT NAME: ______________________________

STUDENT SIGNATURE: ______________________________

Questions

1. (8 points) Evaluate the following definite integral

$$\int_{-1}^{2} \sqrt{4-x^2}\,dx.$$



WARNING: The above picture depicts the graph of $y = \sqrt{4-x^2}$. The integration limit is from -1 to 2, NOT from -2 to 2. Therefore, the definite integral does NOT correspond to the area beneath the entire graph.

A. $\dfrac{4\pi}{3} + \dfrac{1}{2}$

B. $\dfrac{4\pi}{3} + \dfrac{\sqrt{3}}{2}$

C. $\dfrac{4\pi}{3}$

D. $2\pi + \sqrt{3}$

E. 2π

2. (8 points) Compute the following limit:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{n}{n^2+k^2}\right).$$

HINT: Identify the sum as an approximation for a definite integral (i.e., a Riemann Sum).

Then compute the limit as the definite integral.

Use the formula below if necessary:

$$\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \sum_{k=1}^{n}\frac{1}{1+(k/n)^2}\cdot\frac{1}{n}.$$

A. $\dfrac{\ln 2}{2}$

B. $\dfrac{\ln 3}{2}$

C. $\dfrac{\pi}{4}$

D. $\dfrac{\pi}{2}$

E. 3

3. (8 points) Let $F(x) = \displaystyle\int_{x}^{x^2} \sin(\pi t)dt$.

What is the slope of the tangent line to the graph of $y = F(x)$ at $x = \dfrac{1}{2}$?

A. $\dfrac{1}{\sqrt{2}} - 1$

B. $\dfrac{1}{\sqrt{2}}$

C. $\dfrac{\pi}{4}$

D. $\dfrac{\pi}{2} - 1$

E. 1

4. (8 points) Suppose that 80 % of a radioactive substance decays in 5 hours. Find the half-life of the radioactive substance.

A. $\dfrac{8\ln 2}{5\ln 20}$ hours

B. $\dfrac{5\ln 10}{\ln 5}$ hours

C. $\dfrac{\ln 2}{5\ln 5}$ hours

D. $\dfrac{8\ln 5}{\ln 10}$ hours

E. $\dfrac{5\ln 2}{\ln 5}$ hours

5. (8 points) Compute

$$\int_0^{\pi/4} \sqrt{1+\tan t}\cdot(\sec t)^2\ dt.$$

A. $\dfrac{\pi}{2} - 1$

B. $\dfrac{\pi}{4} - 1$

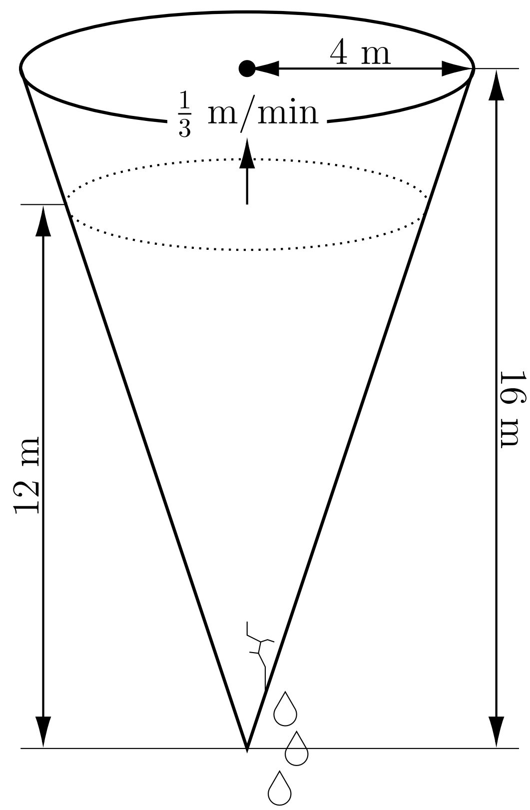
C. $2\sqrt{2} - 2$

D. $\dfrac{2}{3}(2\sqrt{2} - 1)$

E. $\dfrac{2}{3}(\pi - 1)$

6. (8 points) A leaky water tank has the shape of an inverted cone. The tank is 16 m deep and 4 m in radius at the top. Water pours into the tank at a rate of 10 m^3/min. At the moment when the water in the tank is 12 m deep, the water level is increasing at $\frac{1}{3}$ m/min. Assume that the tank leaks at a constant rate.

What is the rate of leakage from the tank ?



HINT: The volume V of an inverted circular cone with radius r at the top and depth h is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

A. 1 m^3/min

B. π m^3/min

C. 3π m^3/min

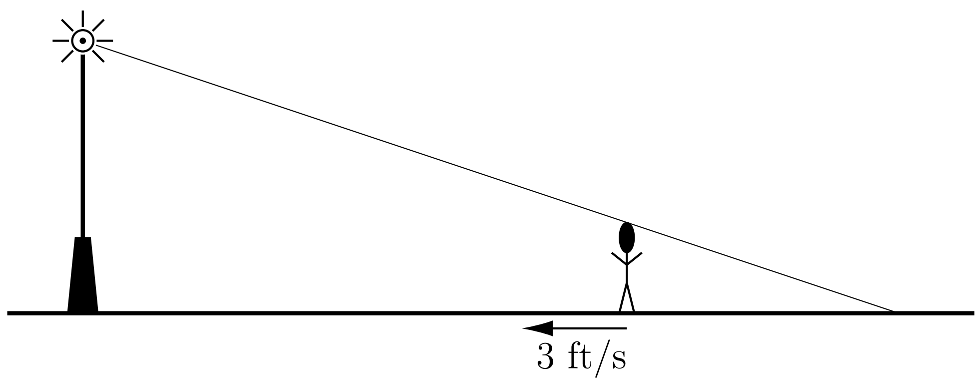
D. $(10 - \pi)$ m^3/min

E. $(10 - 3\pi)$ m^3/min

7. (8 points) A person of 6 feet in height is walking towards a streetlight along a straight path at a rate of 3 feet per second. The height of the streetlight is 18 feet above the ground.

What is the rate in feet/second at which the person's shadow is shortening?

WARNING: We are asking for the rate of change for the length of his shadow, NOT for the rate of change for the distance from the tip of the shadow to the bottom of the streetlight.



A. 3/4

B. 9/4

C. 6

D. 3

E. 3/2

8. (8 points) What is the result of approximating $\sqrt[4]{1.01}$ using the linear approximation of $f(x) = (1 + x^2)^{1/4}$ centered at $a = 0$?

WARNING: We are NOT using the function $g(x) = (1 + x)^{1/4}$ for the linear approximation.

A. 1

B. 1.01

C. 1.005

D. 1.0025

E. 1.001

9. Consider the following function

$$f(x) = |x-2|(x-5)|x-5|.$$

Choose the right statement about the continuity/differentiability of the function

(i) at $x = 2$, and

(ii) at $x = 5$

A. (i) continuous and differentiable
   (ii) continuous and differentiable

B. (i) continuous but not differentiable
   (ii) continuous and differentiable

C. (i) continuous and differentiable
   (ii) continuous but not differentiable

D. (i) not continuous but differentiable
   (ii) continuous and differentiable

E. (i) continuous but not differentiable
   (ii) continuous but not differentiable

10. (8 points) The velocity function, which is the 1st derivative of a position function $f(t)$ is given by the following formula $v = f'(t) = 6(t-1)(t-3)$. Compute the total distance traveled during the first 5 seconds.

A. 40

B. 48

C. 56

D. 80

E. In order to compute the total distance, one has to take the anti-derivative $f(t)$ of $v = f'(t)$. But since the anti-derivative is determined ONLY up to constant, one cannot determine the total distance from the information given above.

11. (8 points) Evaluate the limit

$$\lim_{x\to 0} \frac{\sin(x^2)}{\arctan(\cos(x) - 1)}$$

A. -2

B. -1

C. 0

D. 1

E. 2

12. (8 points) Compute the following limit

$$\lim_{x\to 3^+} \left(\frac{x}{3}\right)^{\frac{6}{x-3}}.$$

A. 1

B. e

C. 2

D. e^2

E. $e^{1/3}$

13. (8 points) Compute the following limit

$$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \left[(2x - \pi) \cdot \tan x\right].$$

A. -2

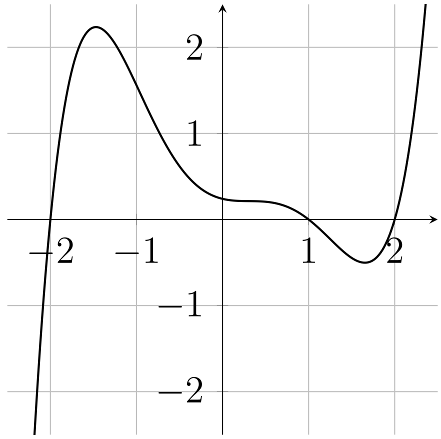
B. 2

C. $-1/2$

D. $1/2$

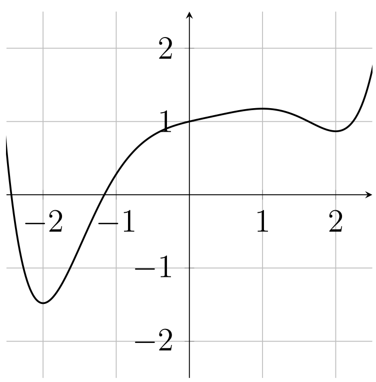
E. ∞

14. (8 points) The derivative $f'(x)$ of a function $f(x)$ has the following graph

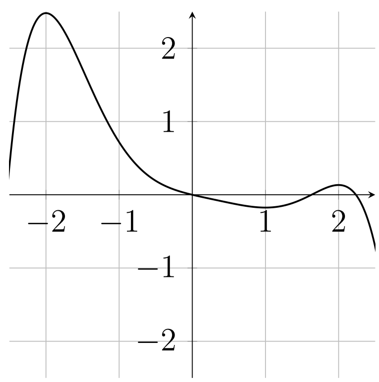


Which one of the following graphs may depict the graph of $f(x)$?

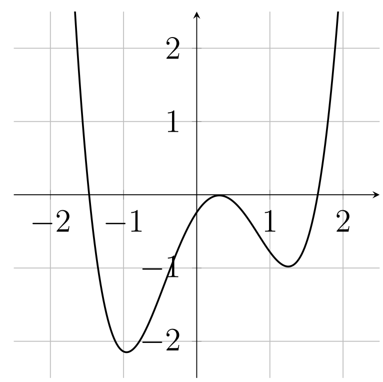
A.



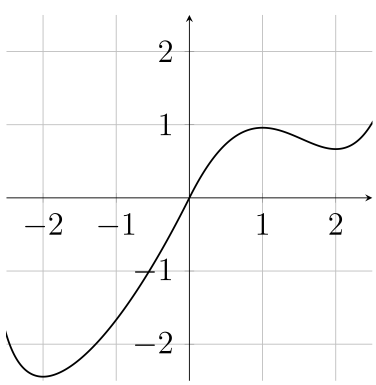
B.



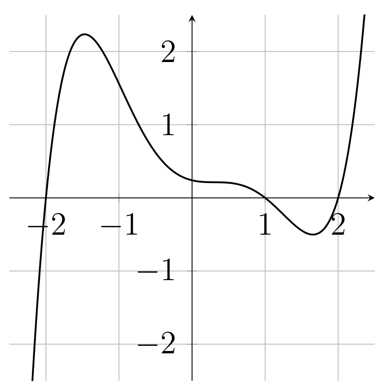
C.



D.

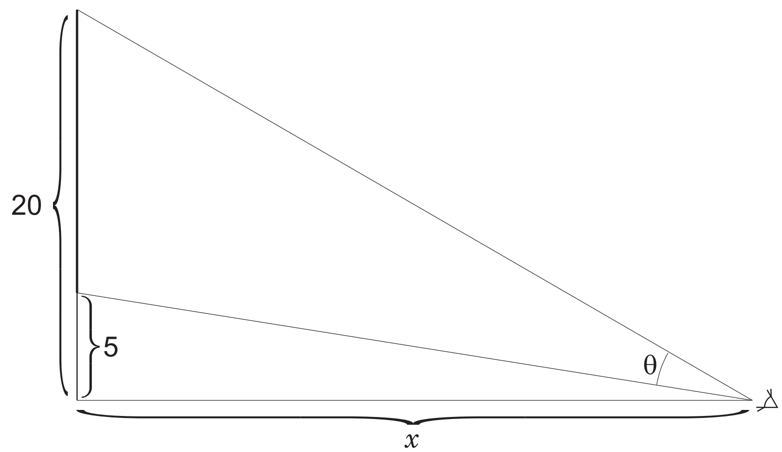


E.



15. (8 points) An auditorium with a flat floor has a large screen on one wall. The lower edge of the screen is 5 feet above eye level and the upper edge of the screen is 20 feet above eye level.

How far from the screen should you stand to maximize your viewing angle ?



HINT: You may use the formula:

$$\theta(x) = \tan^{-1}\left(\frac{20}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right).$$

A. 5 feet away

B. $5\sqrt{2}$ feet away

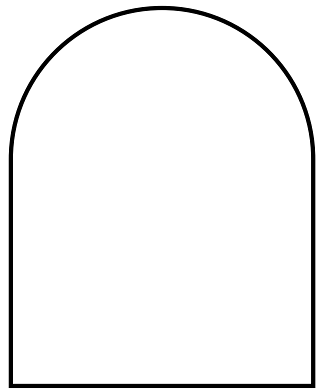
C. 10 feet away

D. $10\sqrt{2}$ feet away

E. 15 feet away

16. (8 points) A *Norman window* is a window in the shape of a rectangle surmounted by a semicircle whose diameter equals the base of the rectangle.

What is the largest possible area of a Norman window with perimeter 4 m ?



A. $\dfrac{4}{4+\pi}$ m^2

B. $\dfrac{8}{4+\pi}$ m^2

C. $\dfrac{4}{4-\pi}$ m^2

D. $4+\dfrac{\pi}{2}$ m^2

E. 2 m^2

17. (8 points) How many solutions are there on the interval $[0, 2\pi]$ for the following equation

$$\cos(2x) = \sin(x) \ ?$$

A. 0

B. 1

C. 2

D. 3

E. 4

18. (8 points) Consider the function $f(x) = x(x^4 - 5)$ over the interval $[a, b] = [0, 2]$.

How many value(s) of c on the interval (a, b) satisfies (satisfy) the statement of the Mean Value Theorem: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

A. 0

B. 1

C. 2

D. 3

E. 4

19. (8 points) Find the equation of the tangent line to the curve

$$x^2 + y^2 + 6x + 3y - 5 = 0$$

at the point $(1, -1)$.

A. $y - 8x + 9 = 0$

B. $y + x = 0$

C. $2y - 4x + 6 = 0$

D. $8y - x - 7 = 0$

E. $y + 8x - 7 = 0$

20. Consider the function

$$f(x) = \frac{-3x+2}{x-1}$$

where its domain is restricted to $(1, \infty)$.

Find

(i) the domain of the inverse function, and

(ii) the formula for the inverse function.

A. (i) $(-\infty, 1)$

(ii) $f^{-1}(x) = \dfrac{1-x}{2-3x}$

B. (i) $(-\infty, 1)$

(ii) $f^{-1}(x) = \dfrac{1-2x}{2-3x}$

C. (i) $(-\infty, 1)$

(ii) $f^{-1}(x) = \dfrac{x+2}{x+3}$

D. (i) $(-\infty, -3)$

(ii) $f^{-1}(x) = \dfrac{x+2}{x+3}$

E. (i) $(-3, \infty)$

(i) (ii) $f^{-1}(x) = \dfrac{x+2}{x+3}$

21. (8 points) Find the real number a so that the following function becomes continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} a & \text{if} \quad x \leq 1 \\ \dfrac{12(\sqrt{x^2+8}-3)}{x^2-1} & \text{if} \quad x > 1. \end{cases}$$

A. 2

B. 3

C. 1

D. $\dfrac{1}{3}$

E. 0

22. (8 points) The derivative of a function $y = f(x)$ is given by

$$f'(x) = (x-2)^5(x+2)^5$$

Find

(i) the x-coodinate(s) of the local minimum (minima), and

(ii) the x-coodinate(s) of the inflection point(s).

A. (i) 2
   (ii) 0

B. (i) -2
   (ii) 0

C. (i) 2
   (ii) $-2, 0, 2$

D. (i) -2
   (ii) $0, 2$

E. (i) $-2, 2$
   (ii) 0

23. Suppose that g is a function with $g'(1) = 3$.

Let f be the function defined by

$$f(x) = g(2\cos^2(x)).$$

Compute $f'(\pi/4)$.

A. -6

B. -3

C. -2

D. 2

E. 3

24. (8 points) Compute the slope of the tangent line to the curve

$$y = x^{\cos(x)}$$

at the point $(\pi, 1/\pi)$.

A. $\dfrac{1}{\pi}$

B. $\dfrac{-1}{\pi^2}$

C. $\dfrac{\ln(\pi)}{\pi}$

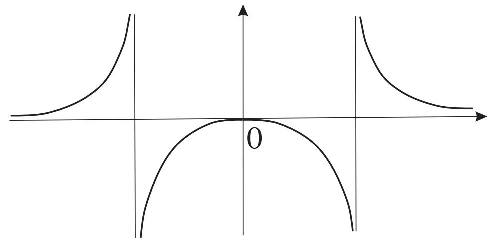
D. $\dfrac{\pi\ln(\pi) - 1}{\pi^2}$

E. $\dfrac{-1}{\pi}$

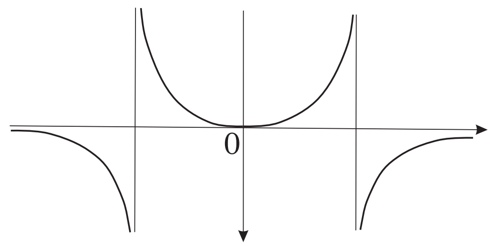
25. (8 points) Which of the following best describes the graph of the function

$$y = f(x) = \frac{x^3}{1 - x^2} \ ?$$

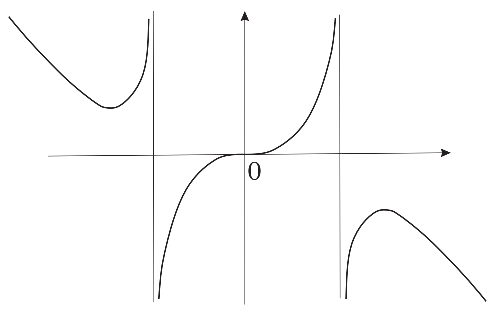
A.



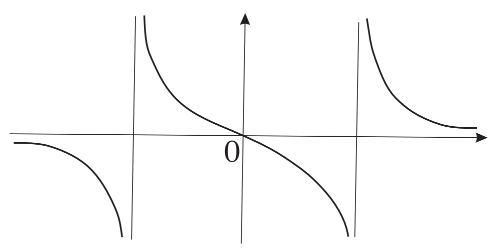
B.



C.



D.



E.