MA 16200: Final Examination

Spring 2024, Purdue University

Exam version: 01

Name: ____________________ PUID #: ____________________

Instruction:

- Follow these instructions carefully. Failure to do so may results in your exam being invalidated and/or an academic integrity violation. All suspected violation of academic integrity will be reported to the Office of the Dean of Students.
- Mark your recitation section below. Write your name and PUID on the top of this cover page. **DO NOT WRITE ANYTHING ELSE** on this cover sheet.

✓	Sec #	Time	TA Name
	0027	7:30AM	Nathan Kapsin
	0050	7:30AM	Brian Wen
	0028	8:30AM	Nathan Kapsin
	0029	8:30AM	Brian Wen
	0048	8:30AM	Sina Nadi
	0052	8:30AM	Ali Sheikh
	0046	8:30AM	Aaron Thomas
	0049	9:30AM	Sina Nadi
	0053	9:30AM	Ali Sheikh
	0047	9:30AM	Aaron Thomas
	0051	10:30AM	Mohit Pandiya
	0032	11:30AM	Mohit Pandiya

✓	Sec #	Time	TA Name
	0016	12:30PM	Tanmay Devale
	0018	12:30PM	Risa Fines
	0023	12:30PM	Cian Nolan
	0015	1:30PM	Tanmay Devale
	0017	1:30PM	Risa Fines
	0024	1:30PM	Cian Nolan
	0031	1:30PM	Mary Collins
	0030	2:30PM	Mary Collins
	0014	2:30PM	Madison Sullivan
	0013	3:30PM	Madison Sullivan
	0025	3:30PM	Conner Partaker
	0026	4:30PM	Conner Partaker

- Use a **#2 PENCIL** to mark the scantron sheet. Fill in the following information:
  - **Your Name:** If there are not enough spaces, fill in as much as you can.
  - **Section Number:** Use all four digits as indicated in the table above.
  - **Test Number:** Fill in 01 for this version of exam.
  - **Student Identification Number:** Fill in your 10-digit PUID with two leading zeros.
  - Write down your TA's name and sign the scantron sheet.
  - Black in your answers in the spaces provided for questions 1–25.
- Do not open the exam booklet or start writing before the proctor signals the start of the exam.
- Do all your work in this exam booklet. Use the back sides of the exam booklet for scratch work.
- Calculators, electronic devices, books, or notes are **NOT ALLOWED**.
- Students may not look at anybody else's exam, and may not communicate with anybody else except with their TA or instructor if there is a question.
- Turn in both the scantron sheet and the exam booklet when you are finished.
- If you finish the exam before 5:25 pm, you may leave the room after turning in the scantron sheet and the exam booklet. You may not leave the room before 3:50 pm. If you don't finish before 5:25 pm, **YOU MUST REMAIN SEATED** until your TA comes and collects your scantron sheet and your exam booklet. You must stop working when the proctor signals the end of exam.

This exam consists of 25 questions. Each question is worth 1 point. You have exactly two hours to finish the exam. Good luck!

Questions:

1. Consider the series

$$\text{(I)} \sum_{k=1}^{\infty} (-1)^k \frac{10^k}{k^{10}} \qquad \text{and} \qquad \text{(II)} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}}$$

   Determine whether each one is absolutely convergent, conditionally convergent, or divergent.

   (A) (I) and (II) are both conditionally convergent.
   (B) (I) is divergent; (II) is conditionally convergent.
   (C) (I) and (II) are both absolutely convergent.
   (D) (I) is conditionally convergent; (II) is absolutely convergent.
   (E) (I) is divergent; (II) is absolutely convergent.

2. What is the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \left(1 + \frac{2}{k}\right)^k x^k \quad ?$$

   (A) 1
   (B) e^2
   (C) 2
   (D) e^{-2}
   (E) 0

3. What are the first three (3) terms of the Maclaurin series for $f(x) = \sqrt{1+x}$?

(A) $1 - \frac{x}{2} + \frac{x^2}{8}$

(B) $1 + \frac{x}{2} - \frac{x^2}{8}$

(C) $1 + \frac{x}{2} + \frac{x^2}{8}$

(D) $-1 + \frac{x}{2} - \frac{x^2}{8}$

(E) $-1 - \frac{x}{2} - \frac{x^2}{8}$

4. Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^n(2n+1)}$$

Hint: Use the Maclaurin series for $\arctan(x)$.

(A) $\frac{\sqrt{3}\pi}{6}$

(B) $\frac{\pi}{6}$

(C) $\frac{\sqrt{3}\pi}{3}$

(D) $\frac{\pi^2}{6}$

(E) $\frac{\pi}{3}$

5. Two vectors $\vec{v}$ and $\vec{w}$ in three dimensions satisfy that

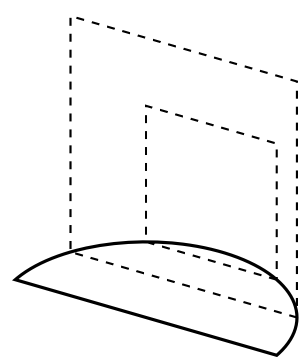
$$\vec{v} \cdot \vec{w} = -3 \quad \text{and} \quad \vec{v} \times \vec{w} = \langle 1, -2, 2 \rangle$$

What is the angle between $\vec{v}$ and $\vec{w}$?

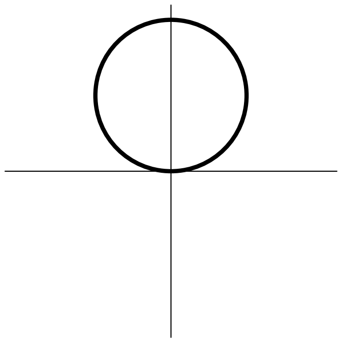
(A) $\pi/4$
(B) $2\pi/3$
(C) $\pi/3$
(D) $3\pi/4$
(E) $\pi/6$

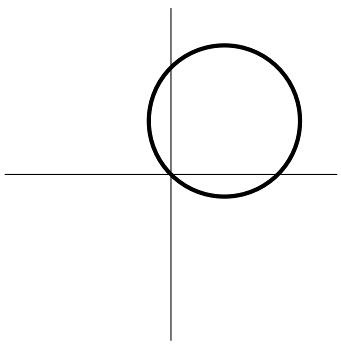
6. A solid has a semicircular base of radius 2. Cross sections perpendicular to the base and parallel to the straight edge of the semicircle are squares. What is the volume of the solid?

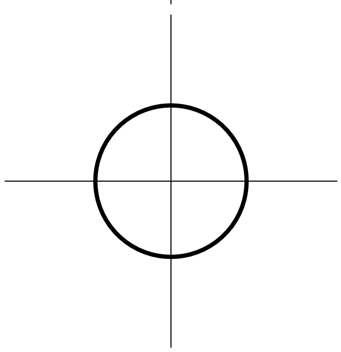
(A) $16/3$
(B) 16π
(C) 6π
(D) $3\pi/2$
(E) $64/3$

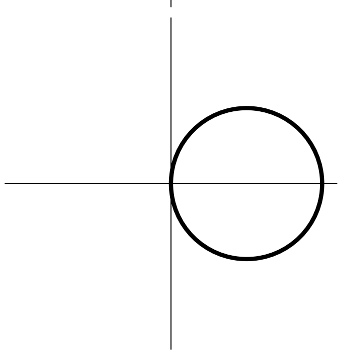


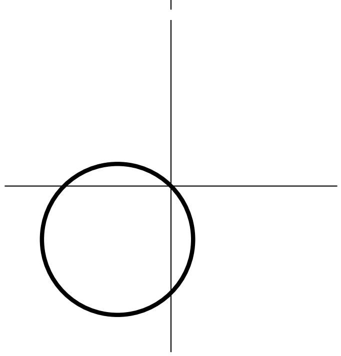
7. Which of the following shows the graph of the polar curve $r = \cos(\theta) + \sin(\theta)$ relative to the coordinate axes?

(A) 

(B) 

(C) 

(D) 

(E) 

8. Find the value of A if

$$\int_0^{\pi/2} e^{2t}\sin(t)\,dt = A + \int_0^{\pi/2} 2e^{2t}\cos(t)\,dt.$$

(A) -1
(B) $1/2$
(C) 2
(D) $-e^{\pi}$
(E) 1

9. If the power series $\sum_{k=1}^{\infty} c_k(x-1)^k$ converges at $x = -1$ and diverges at $x = 4$ what can be said about its radius of convergence R?

(A) $2 \leq R \leq 3$
(B) $2 < R < 3$
(C) $1 \leq R \leq 2$
(D) $2 \leq R < 3$
(E) $1 \leq R \leq 4$

10. Convert the polar coordinates $(4, 7\pi/2)$ to Cartesian coordinates.

(A) $(-2\sqrt{2}, 2\sqrt{2})$
(B) $(2\sqrt{2}, -2\sqrt{2})$
(C) $(2\sqrt{3}, 2)$
(D) $(0, -4)$
(E) $(0, 4)$

11. If the Maclaurin series of the function $f(x)$ is

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{3n(n+6)},$$

then $f^{(6)}(0)$ is equal to

(A) $8/5$
(B) $5/3$
(C) $10/3$
(D) $9/7$
(E) $-15/6$

12. Which one of the following power series equals $f(x) = \dfrac{1}{(x-2)^2}$ near the origin?

(A) $\displaystyle\sum_{k=1}^{\infty} 2kx^{k-1}$

(B) $\displaystyle\sum_{k=1}^{\infty} \frac{kx^{k-1}}{2^k}$

(C) $\displaystyle\sum_{k=1}^{\infty} (-1)^k 2kx^{k-1}$

(D) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k kx^{k-1}}{2^{k+1}}$

(E) $\displaystyle\sum_{k=1}^{\infty} \frac{kx^{k-1}}{2^{k+1}}$

13. If

$$\sum_{k=0}^{\infty} c_k (x-2)^k$$

is the Taylor series for $f(x) = \ln(3x)$ centered at $a = 2$, what is the value of c_3?

(A) $-1/4$

(B) $1/4$

(C) $1/8$

(D) $-1/24$

(E) $1/24$

14. Let $a_n = \dfrac{\sin(n)}{n^2}$, $b_n = \sin(n\pi/2)$, and $c_n = n\sin(1/n)$ be sequences. Determine whether each sequence is convergent or divergent.

(A) a_n is convergent; b_n and c_n are divergent.

(B) a_n is divergent; b_n and c_n are convergent.

(C) a_n is convergent; b_n is divergent; c_n is convergent.

(D) All three sequences are convergent.

(E) a_n and b_n are convergent; c_n is divergent.

15. Which type of curve does the polar equation $r = 4\sec(\theta)$ describe in polar coordinates?

(A) A heart-shaped curve (cardioid)

(B) An ellipse

(C) A parabola

(D) A circle

(E) A line

16. Find the arc length of $r = e^{2\theta}$, where $0 \leq \theta \leq 2\pi$.

(A) $\frac{\sqrt{9}}{2}(e^{4\pi} - 1)$

(B) $\frac{\sqrt{5}}{2}(e^{4\pi} - 1)$

(C) $\sqrt{2}(e^{4\pi} - 1)$

(D) $\frac{\sqrt{2}}{2}(e^{4\pi} - 1)$

(E) $\frac{\sqrt{3}}{2}(e^{4\pi} - 1)$

17. What are all values of p that will make the following series converge?

$$\sum_{n=3}^{\infty} \frac{(\ln(n))^p}{n}$$

(A) $p \geq -1$

(B) $p > -1$

(C) $p < -1$

(D) There is no value of p that makes the series converge.

(E) p can be any real number.

18. What is the area between the spirals $r = \theta$ and $r = 2\theta$, with $0 \leq \theta \leq \pi/4$?

(A) $\pi^3/64$
(B) $\pi^3/128$
(C) $\pi^3/16$
(D) $\pi^3/32$
(E) $\pi^3/4$

19. What is the interval of convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{n+5} \quad ?$$

(A) $(1, 3]$
(B) $[1, 3)$
(C) $(1, 3)$
(D) $[0, 3]$
(E) $[1, 3]$

20. Evaluate the integral

$$\int_1^{\infty} \frac{1}{x^2 + x^4}\, dx$$

(A) $1 - \frac{\pi}{2}$
(B) $1/3$
(C) $1 - \frac{\pi}{4}$
(D) $4/3$
(E) The integral does not converge.

21. The integral

$$\int \frac{e^{2x} - 1}{x}\, dx$$

is equal to

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1} x^n}{n! \cdot n} + c$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n!} + c$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n} + c$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^{n+1}}{(n+1)!} + c$

(E) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n! \cdot n} + c$

22. Which one of the following is the most appropriate substitution for the integral

$$\int \sqrt{x^2 - 4x + 3}\, dx \quad ?$$

(A) $x = \sin(\theta) + 2$
(B) $x = \cot(\theta) + 2$
(C) $x = \sec(\theta) + 2$
(D) $x = \tan(\theta) + 2$
(E) $x = \cos(\theta) + 2$

23. Which of the following polar coordinates describe(s) the point $(2\sqrt{3}, 2)$ in Cartesian coordinates?

(I) $(4, \pi/6)$
(II) $(-4, -11\pi/6)$
(III) $(-4, 7\pi/6)$

(A) All of (I), (II), and (III)
(B) Only (I) and (III)
(C) Only (II)
(D) Only (III)
(E) Only (I)

24. A cubic tank whose sides are 1 m long sits on the ground and is filled with a liquid of density 1200 kg/m^3. If we take gravitational acceleration $g = 10$ m/s^2, compute the work (in joules) neccessary to empty the tank by pumping the liquid through its top.

(A) 6000

(B) 8000

(C) 3000

(D) 5000

(E) 4000

25. Given that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},$$

find the smallest number of terms of the series we need to add to compute the value of $\ln(1.1)$ with an error of less than or equal to 10^{-8}.

(A) 2

(B) 5

(C) 9

(D) 7

(E) 3

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Common Maclaurin series:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } -\infty < x < \infty \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } -\infty < x < \infty \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } -\infty < x < \infty \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1 \\
\arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } -1 \le x \le 1
\end{aligned}$$

Common trigonometric identities:

$$\begin{aligned}
&\cos(x)^2 + \sin(x)^2 = 1 \\
&\sec(x)^2 - \tan(x)^2 = 1 \\
&\sin(2x) = 2\cos(x)\sin(x) \\
&\cos(2x) = \cos(x)^2 - \sin(x)^2 \\
&\cos(2x) = 2\cos(x)^2 - 1 \\
&\cos(2x) = 1 - 2\sin(x)^2
\end{aligned}$$