# Asymptotic flatness of Morrey extremals

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(joint work with Frank Seuffert)

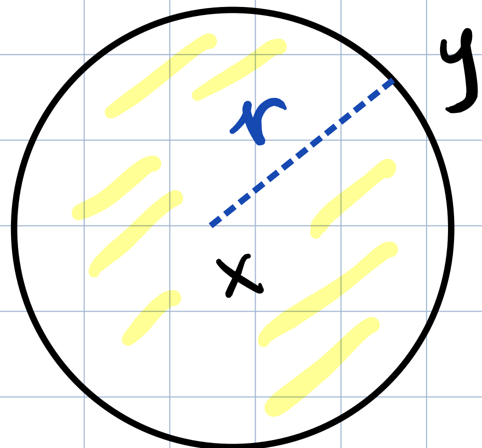
## Morrey's estimate $p>n$

$$\boxed{|u(x)-u(y)| \le C\, r^{1-n/p}\left(\int_{B_r(z)} |Du|^p\right)^{1/p}}$$

$x, y \in B_r(z)$. Here

$$\boxed{C = C(n,p)}$$

- $x, y \in \mathbb{R}^n \qquad r = |x-y|, \qquad y \in \overline{B_r(x)}$




$$\boxed{|u(x) - u(y)| \leq C\,|x-y|^{1-n/p} \left( \int_{B_r(x)} |Du|^p \right)^{1/p}}$$

## Morrey's Inequality $\quad p > n$

$$\boxed{\sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{1/p}}$$

or

$$\boxed{[u]_{1-n/p} \leq C \|Du\|_p .}$$

Questions: What is the best constant $C$? What are the non-trivial extremals $u$?

$$\boxed{[u]_{1-n/p} = C \|Du\|_p}$$

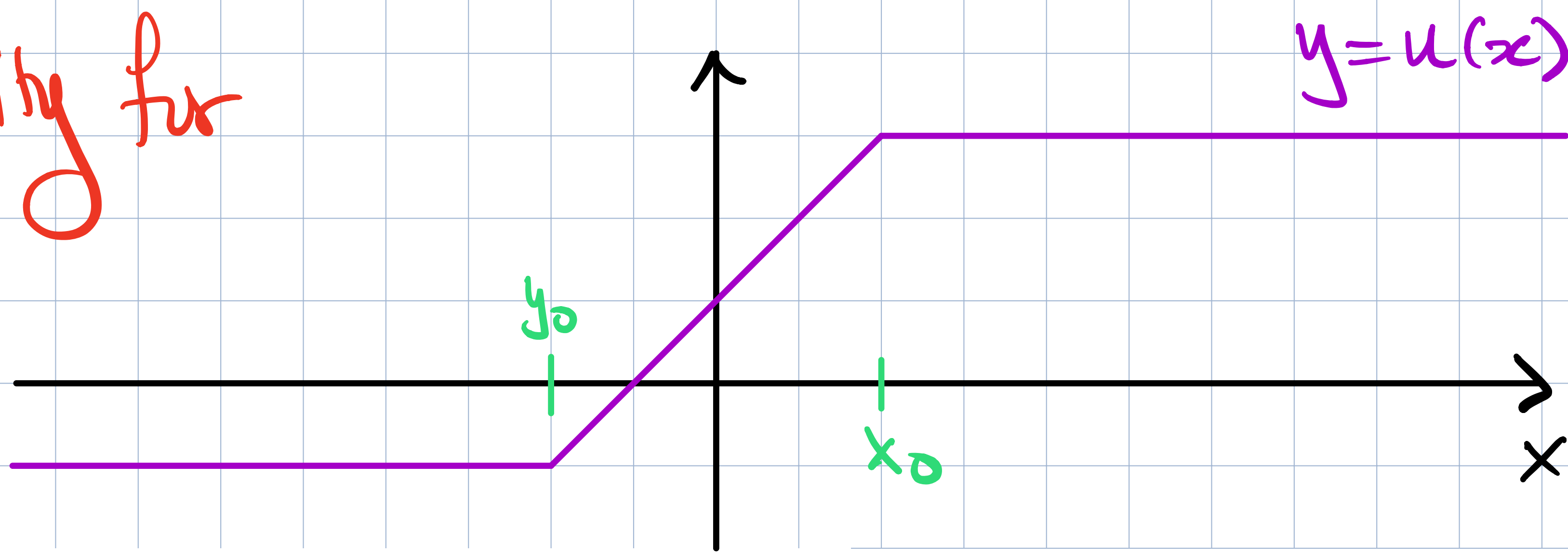
Both questions are unresolved $(n \geq 2)$.

$\underline{n=1}$ : $C=1$

$$\left| u(x) - u(y) \right| = \left| \int_y^x u'(t)\,dx \right| \leq |x-y|^{1-1/p} \, \|u'\|_p$$

$$\boxed{[u]_{1-1/p} \leq \|u'\|_p}$$

Equality for



- Suppose there is an extremal $u$ with

$$\boxed{[u]_{1-n/p} = \frac{|u(x_0)-u(y_0)|}{|x_0-y_0|^{1-n/p}}}$$

Note

$$\boxed{\frac{|u(x_0)-u(y_0)|^p}{|x_0-y_0|^{p-n}} = C^p \int_{\mathbb{R}^n} |Du|^p dx.}$$

Then for $t \in \mathbb{R}$, $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\boxed{\frac{\left| u(x_0) + t\phi(x_0) - \left(u(y_0) + t\phi(y_0)\right)\right|^p}{|x_0 - y_0|^{p-n}} \leq C^p \int_{\mathbb{R}^n} |Du + tD\phi|^p \, dx}$$

$$\boxed{\frac{|u(x_0) - u(y_0)|^p}{|x_0 - y_0|^{p-n}} = C^p \int_{\mathbb{R}^n} |Du|^p \, dx.}$$

- Subtracting, dividing by $t>0$, sending $t\to 0^+$ gives

$$\boxed{\frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{|x_0-y_0|^{p-n}}(\phi(x_0)-\phi(y_0)) \le C^p\int_{\mathbb{R}^n}|Du|^{p-2}Du\cdot D\phi\,dx}$$

Repeating the computation with $t \to 0^-$ gives

$$\boxed{\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D\phi \, dx = c\left(\phi(x_0) - \phi(y_0)\right)}$$

for some $c \in \mathbb{R}$. That is

$$\boxed{-\Delta_p u = c\left(\delta_{x_0} - \delta_{y_0}\right)}$$

in $\mathbb{R}^n$. Here $\Delta_p w = \operatorname{div}\left(|Dw|^{p-2} Dw\right)$.

**Theorem**: TFAE

(i) $u$ extremal with

$$\boxed{[u]_{1-n/p} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}}$$

(ii) $\boxed{-\Delta_p u = c(\delta_{x_0} - \delta_{y_0})}$ in $\mathbb{R}^n$

(iii) $\boxed{\int_{\mathbb{R}^n} |Du|^p dx \leq \int_{\mathbb{R}^n} |Dv|^p dx}$ whenever

$u(x_0) = v(x_0)$ & $u(y_0) = v(y_0)$

Corollary: For each distinct $\alpha, \beta \in \mathbb{R}$, $x_0, y_0 \in \mathbb{R}^n$, there is a unique extremal $u$ with

$$\boxed{[u]_{1-n/p} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}}$$

and which minimizes

$$\boxed{\int_{\mathbb{R}^n} |Dv|^p dx}$$

subject to $v(x_0) = \alpha$, $v(y_0) = \beta$.

**Corollary**: $u$ extremal with $\boxed{u(\pm e_n)=\pm 1}$

and $$\boxed{[u]_{1-n/p}=\frac{|u(e_n)-u(-e_n)|}{|e_n-(-e_n)|^{1-n/p}}}.$$

(i) $u(\theta x)=u(x)$, $\theta^t\theta=I_n$ $\theta e_n=e_n$

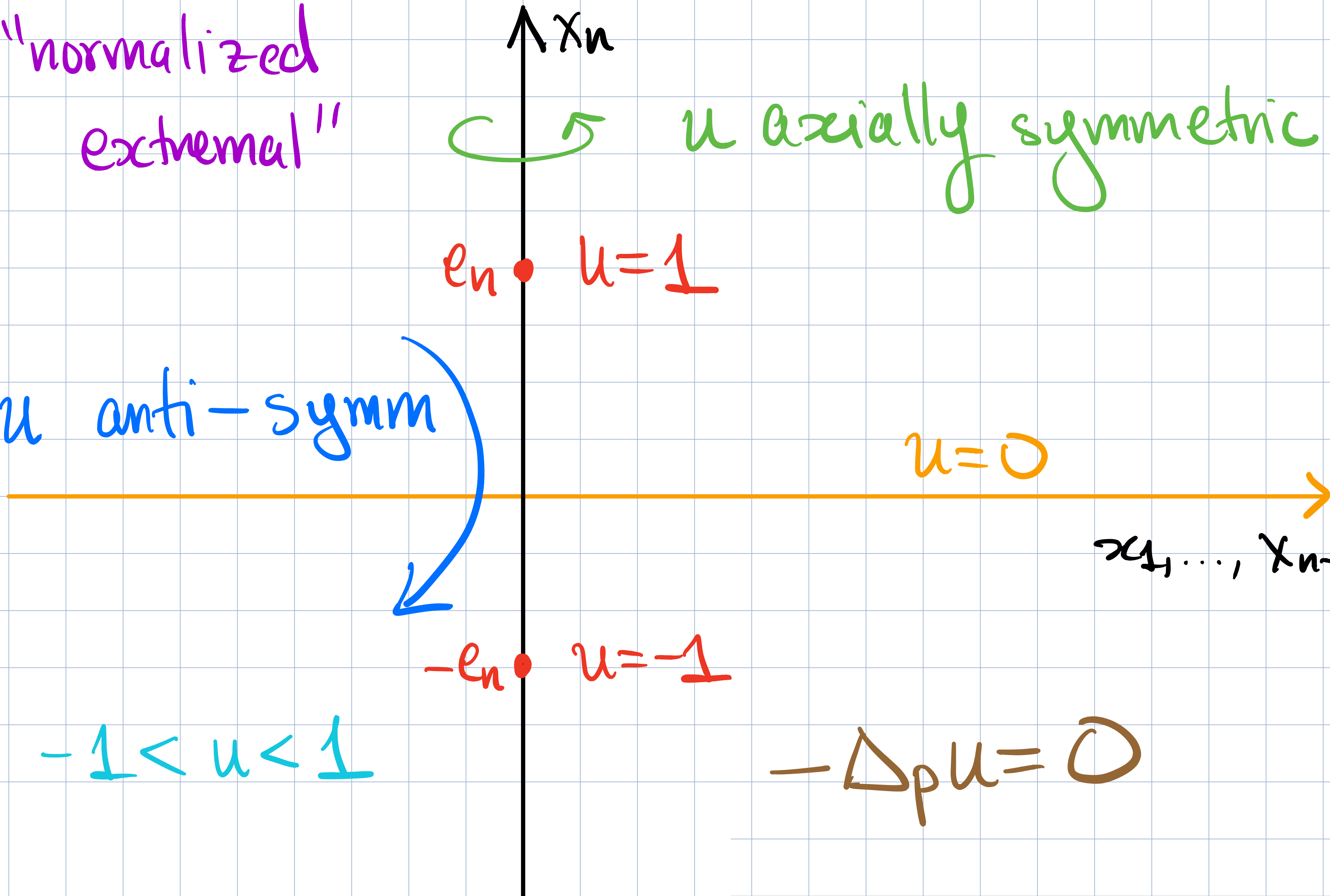
(ii) $u(x-2x_ne_n)=-u(x)$

(iii) $-1\leq u(x)\leq 1$

**Remark**: $\boxed{u(x)=0 \quad \text{on} \quad x_n=0}.$

"normalized extremal"

$u$ axially symmetric



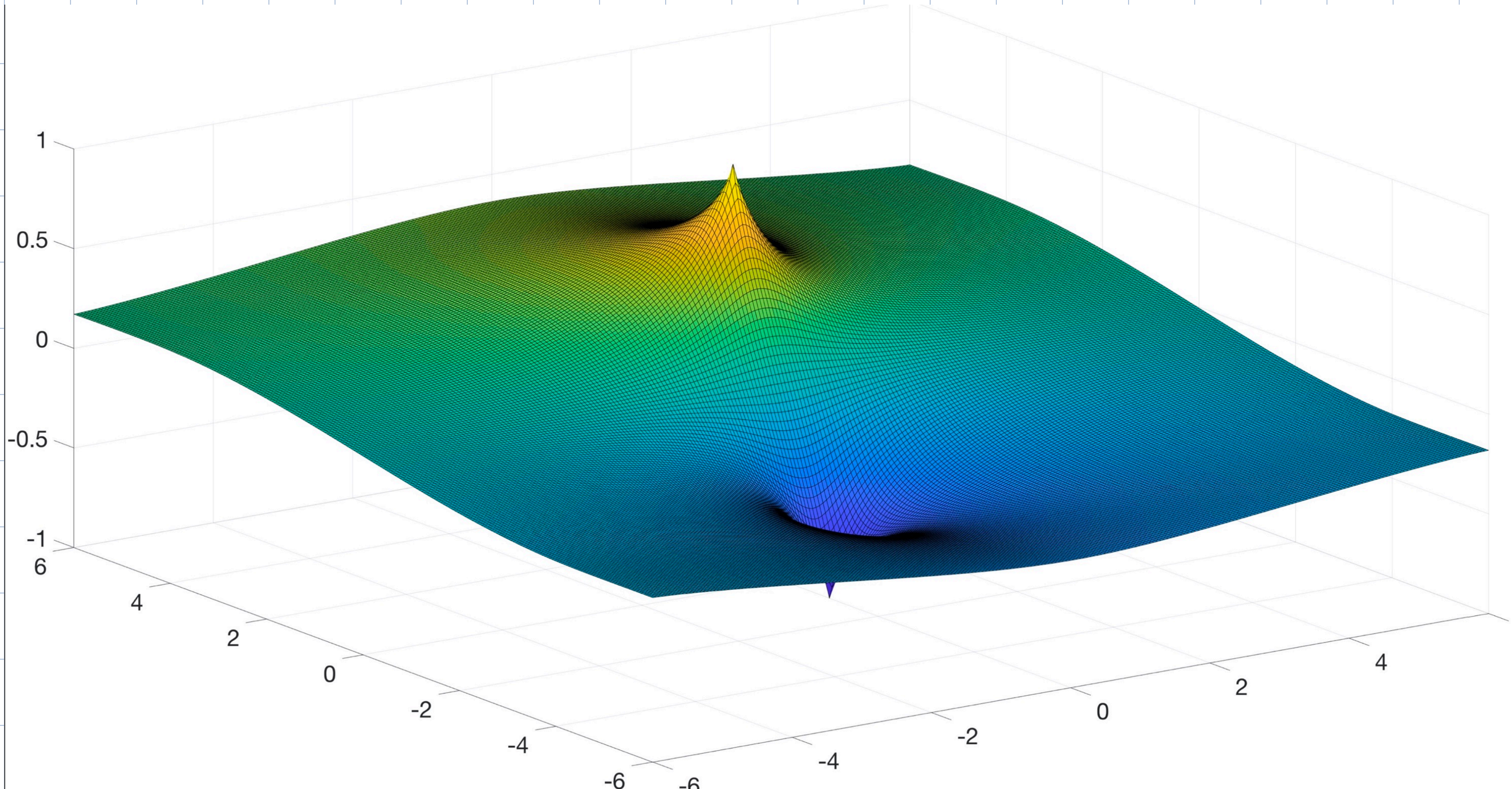

$u$ anti-symm

$-1 < u < 1$

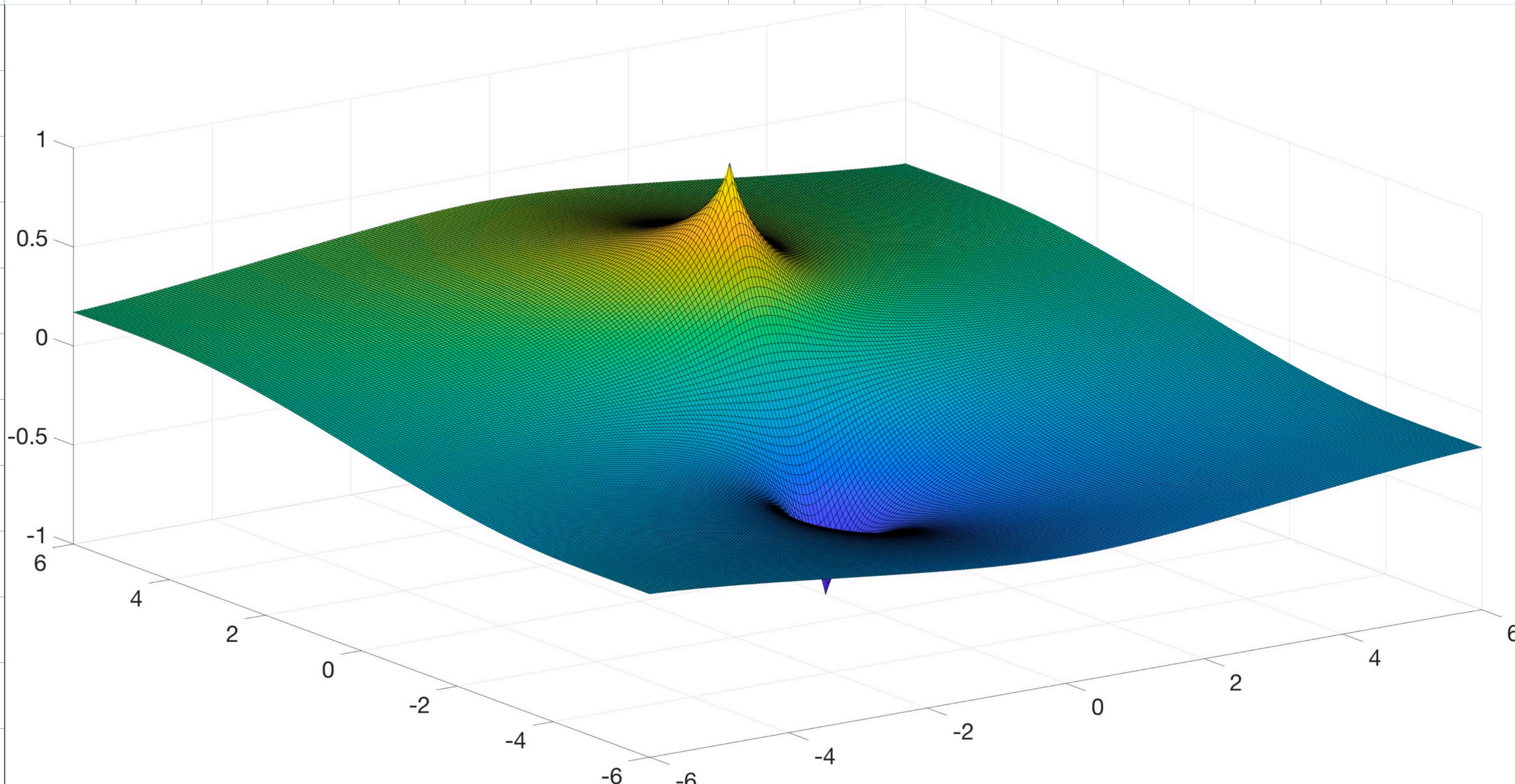
$-\Delta_p u = 0$

Numerical Computation: $n=2,\ p=4,\ x_0=e_2$

$y_0=-e_2 \qquad u(\pm e_2)=\pm 1.$



Natural question: what is the behavior of $u(x)$ as $|x| \to \infty$ ?



**Theorem:** $$\boxed{\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} |x|\,|Du(x)| = 0}$$

**Key Lemma:** $v \in W^{1,p}_{loc}(\mathbb{R}^n)$ satisfies

$$\boxed{-\Delta_p v = c\,\delta_{x_0}}$$

some $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $v \equiv \text{const}$ & $c = 0$.

Sketch of Proof (Theorem): 1. We'll use that $|u(y)| \leq 1$ all $y \in \mathbb{R}^n$. For $t>0$,

$$\boxed{v_t(x) := u(tx)}$$

for $|x| \geq 1/t$. Note $|v_t(x)| \leq 1$ and

$$\boxed{-\Delta_p v_t = 0 \quad \text{in } |x| > 1/t}$$

since $-\Delta_p u = c(\delta_{e_n} - \delta_{-e_n})$ in $\mathbb{R}^n$.

$C^{1,\alpha}_{loc}$ estimates (Uraltseva, Lewis, Evans)

$$\boxed{\| v_t \|_{C^{1,\alpha}(K)} \leq A}$$

for $K \subseteq \mathbb{R}^n \setminus \{0\}$ compact $t$ large

Thus $v_{t_k} \to v_\infty$ in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$

$$\boxed{-\Delta_p v_\infty = 0 \quad \mathbb{R}^n \setminus \{0\}}$$

2. Old results on p-harmonic functions on punctured domains give $v_\infty \in W^{1,p}_{loc}(\mathbb{R}^n)$

$$\boxed{-\Delta_p v_\infty = c\,\delta_{x_0}}$$

Thus

$$\boxed{\lim_{k\to\infty} v_{t_k} = \text{const} = a}$$

in $C^1_{loc}(\mathbb{R}^n\setminus\{0\})$.

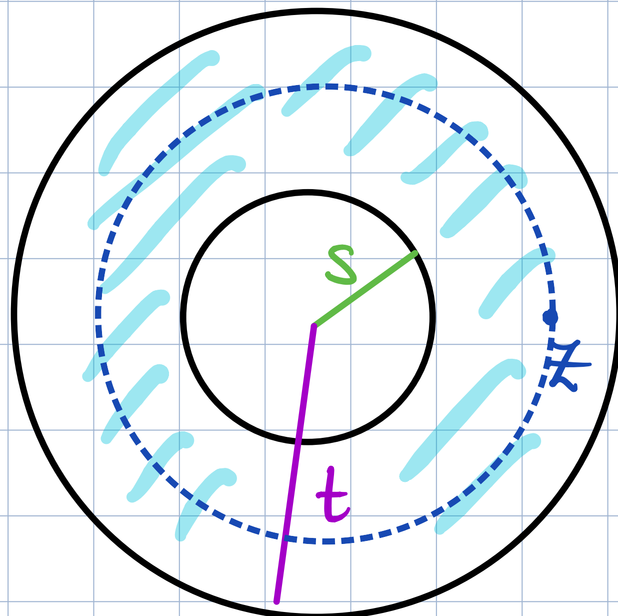
3. Consider

$$\boxed{m(t) = \min_{|y|=t} u(y)}$$

for $t > 1$. By comparison

$$\boxed{u(z) \geq \min\{m(t), m(s)\}}$$

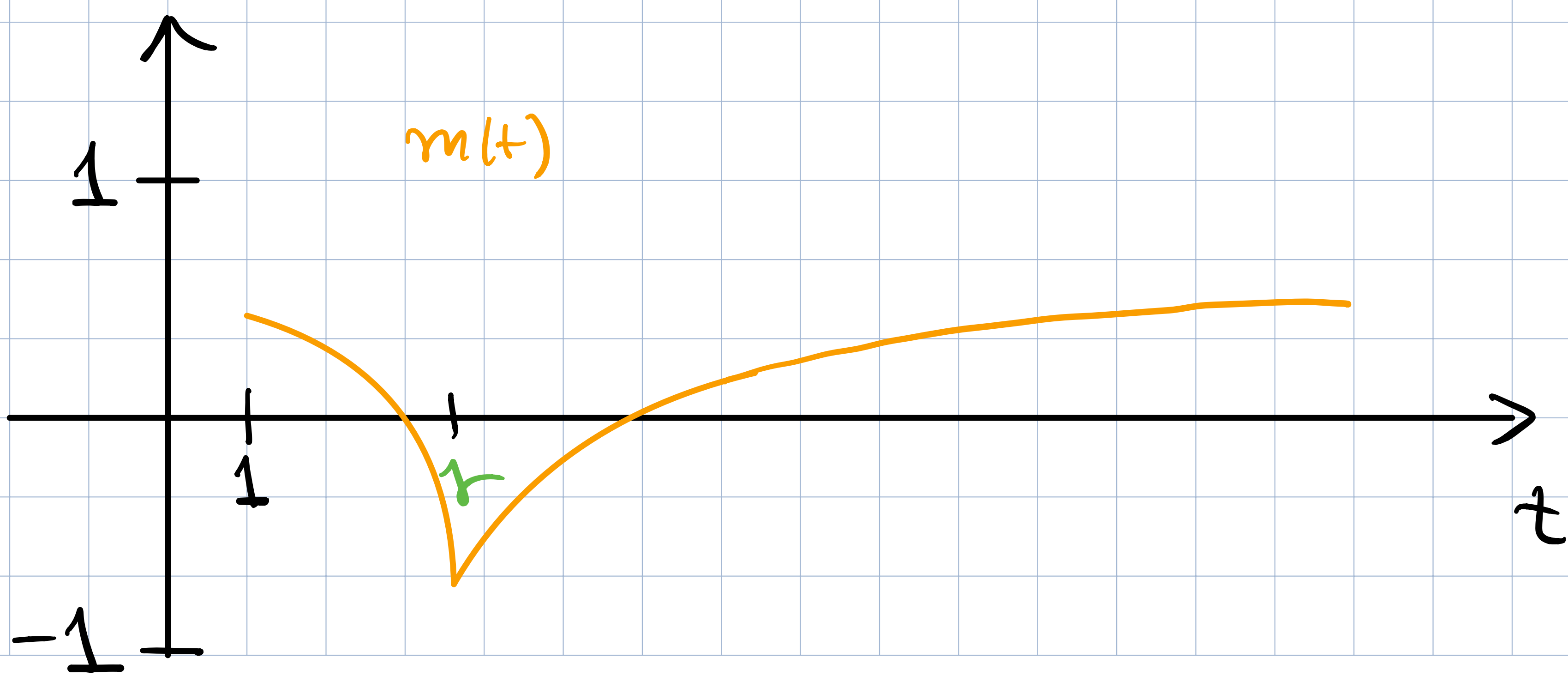
for $1 < s < |z| < t$.



It follows that for $0 \leq \lambda \leq 1$

$$\boxed{m(\lambda s + (1-\lambda)t) \geq \min\{m(s), m(t)\}}$$

$t, s > 1$. Thus $m$ is monotone on $[r, \infty]$



4. We can now choose $|x_{t_k}| = 1$ such that

$$\boxed{\begin{aligned} \lim_{t\to\infty} m(t) &= \lim_{t\to\infty} \min_{|y|=t} u(y) \\ &= \lim_{t\to\infty} \min_{|x|=1} v_t(x) \\ &= \lim_{k\to\infty} v_{t_k}(x_{t_k}) \\ &= a. \end{aligned}}$$

$v_{t_k}(x) \longrightarrow a$ uniformly on $|x|=1$

Likewise

$$\boxed{\lim_{t\to\infty} \max_{|y|=t} u(y) = \lim_{t\to\infty} \max_{|x|=1} v_t(x) = a}$$

and so

$$\boxed{\lim_{|y|\to\infty} u(y) = a .}$$

As $u(y)=0$ on $\{y_n=0\}$, $\underline{a=0}$.

5. $Dv_t(x) = t\,Du(tx) \to 0$ uniformly in $|x|=1$.

so

$$\boxed{|y|\,|Du(y)| = \Big|Dv_{\underbrace{|y|}_{t}}\Big(\frac{y}{|y|}\Big)\Big| \to 0}$$

as $|y| \to \infty$.

Summary: $\boxed{\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} |x|\,|Du(x)| = 0}$

- Remark: These methods also apply to bounded, p-harmonic functions on exterior domains

$$\boxed{\begin{aligned} &-\Delta_p v = 0 \qquad |x|>1 \\ &|v(x)| \le 1 \end{aligned}}$$

In particular they can be used to establish the following Harnack ineq.

**Proposition:** There is $C \geq 1$ such that

$$\boxed{\sup_{|x| \geq 2r} v(x) \leq C \inf_{|x| \geq 2r} v(x)}$$

for each non-negative bounded p-harmonic function $v$ on $|x| \geq r$.

**Theorem**: There is $\alpha$ and $A>0$ such that

$$\boxed{|u(x)| \leq \frac{A}{|x|^\alpha}, \quad |x| \geq 1.}$$

**Key Lemma**: $f : [1,\infty) \to [0,\infty)$ decreasing and $f(2r) \leq \mu f(r)$ some $\mu \in (0,1)$. Then

$$\boxed{f(r) \leq \frac{1}{\mu} r^{\frac{\ln \mu}{\ln 2}} f(1)}$$

## Proof of Theorem: $r \geq 1$.

$$\boxed{M(r) := \sup_{|x| \geq 2r} u(x), \qquad m(r) := \inf_{|x| \geq 2r} u(x)}$$

$$\begin{cases} M(2r) - m(r) = \sup_{|x| \geq 2r} \left( u(x) - m(r) \right) \\ \leq C \inf_{|x| \geq 2r} \left( u(x) - m(r) \right) \\ = C \left( m(2r) - m(r) \right) \end{cases}$$

Likewise

$$M(r) - m(2r) = \sup_{|x| \geq 2r} \{ M(r) - u(x) \}$$

$$\leq C \inf_{|x| \geq 2r} \{ M(r) - u(x) \}$$

$$= C \left( M(r) - M(2r) \right)$$

Combining these inequalities give

$$\boxed{\left( M(2r) - m(2r) \right) \leq \frac{C-1}{C+1} \left( M(r) - m(r) \right).}$$

By the lemma,

$$\boxed{\begin{aligned} |u(x)-u(y)| &\leq M(r)-m(r) \\ &\leq \frac{B}{r^{\alpha}}\, M(1)-m(1) \\ &\leq \frac{A}{r^{\alpha}} \quad (\text{as } |u| \leq 1) \end{aligned}}$$

for $|x|, |y| \geq r$.

$$\boxed{|u(x)| \leq \frac{A}{|x|^\alpha}, \quad |x| \geq 1.}$$

Corollary: There is a constant $B$ such that

$$\boxed{|x|\,|Du(x)| \leq \frac{B}{|x|^\alpha}}$$

for all $|x| > 1$.

**Question:** what is the exact asymptotics of $u(x)$ as $|x| \to \infty$ ?

Thank You!!