On maximal regularity for viscous Hamilton-Jacobi equations

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February 28, 2021

joint works with A. Goffi (Padova)

L^q-maximal regularity

For $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, \quad \gamma > 1, \quad q \geq 1,$ is it true that

$$-\Delta u + |Du|^\gamma \in L^q \quad \Rightarrow \quad \Delta u,\ |Du|^\gamma \in L^q \quad \textbf{?}$$

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For $u : \Omega \times (0, T) \to \mathbb{R}, \quad \gamma > 1, \quad q \geq 1,$ is it true that

$$\partial_t u - \Delta u + |Du|^\gamma \in L^q_{x,t} \quad \Rightarrow \quad \partial_t u,\ \Delta u,\ |Du|^\gamma \in L^q_{x,t} \quad \textbf{?}$$

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- Nonlinear Gagliardo-Nirenberg inequalities ($\gamma = 2$)
- Conjectured by P.-L. Lions $\sim$ '12-'14 to hold iff

$$q > d\frac{\gamma - 1}{\gamma} \quad =: \bar{q}_{d,\gamma}$$

(in the stationary setting)

On the exponent $\bar{q}_{d,\gamma}$

- Gain of regularity: $-\Delta u = -|Du|^\gamma + f$

$$\|u\|_{W^{2,q}} \lesssim \| \, |Du|^\gamma \, \|_{L^q} + \|f\|_{L^q} \qquad \text{Calderón-Zygmund}$$

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and

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- Scaling properties of the equation
- $q \geq \bar{q}_{d,\gamma}$ is necessary for general existence of weak solutions [Hansson, Maz'ya, Verbitsky '99]
- FALSE whenever $q < \bar{q}_{d,\gamma}$:
  consider suitable truncations of $v(x) \sim |x|^{\frac{\gamma-2}{\gamma-1}}$, which satisfies $-\Delta v + |Dv|^\gamma = 0$.

Sub-quadratic vs Super-quadratic

"For Hamilton-Jacobi equations with second order diffusion having at most quadratic growth in the first order term, the second order diffusion dominates at small scales. "

The growth $\gamma = 2$ is usually referred as the *natural growth*:

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- Euler-Lagrange equation for quadratic functionals

$$\min \iint a(u)\frac{|Du|^2}{2} \qquad -\mathrm{div}(a(u)Du) + \frac{1}{2}a'(u)|Du|^2 = 0$$

- "invariance" through the chain rule

$$-\Delta u + |Du|^2 = 0 \quad \overset{u=\varphi(v)}{\qquad} \quad -\Delta v + \left(\varphi' - \frac{\varphi''}{\varphi'}\right)|Dv|^2 = 0$$

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- Hopf-Cole transform

$$-\Delta u + |Du|^2 = f \quad \overset{u=-\log w}{\qquad} \quad -\Delta w + wf = 0 \ .$$

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- General solvability with bounded and unbounded ingredients;
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- General solvability with bounded and unbounded ingredients;
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When $\gamma > 2$, some "unnatural" phenomena occur, e.g.

- Sub-solutions are Hölder continuous with bounded and unbounded f
- Hölder continuity extends up to the boundary

Nonlinear G-N

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$$-\Delta u + |Du|^2 = f$$

and one has

$$\| \, |Du|^2 \, \|_{L^q} \leq C = C(\, \|f\|_{L^q} \,),$$

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since $w = e^{-u}$ satisfies $-\Delta w + wf = 0$, we obtain

$$\int \frac{|Dw|^{2q}}{w^{2q}} = \int \Bigg(|Du|^{2q} = \| \, |Du|^2 \, \|^q_{L^q} \leq C(\, \|f\|_{L^q} \,) \sim C \quad \int \frac{|\Delta w|^q}{w^q} \Bigg) \Bigg($$

The inequality can be proven using (linear) L^q-max. regularity and the Harnack inequality - for $q > \frac{d}{2} = \bar{q}_{d,2}$.

Literature

- Pioneering works [Serrin, Ladyzhenskaja, Amann, Crandall, Lions, ...] : strong solutions / weak (energy) solutions, $\gamma \leq 2$

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- Symmetrization techniques [Alvino, Ferone, Trombetti, ...]
- Nonlinear potential theory [Maz'ya, Phuc, ...] , $\gamma > 1$
- Viscosity solutions: $f \in C$ or $W^{1,\infty}$.

Bernstein's method

We need a genuinely nonlinear method.

Setting $v = |Du|^2$, v solves

$$-\Delta v + \gamma |Du|^{\gamma-2} Du \cdot Dv + |D^2 u|^2 = Df \cdot Du.$$

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Equation can be plugged in

$$|D^2 u|^2 \geq |\Delta u|^2 = (|Du|^\gamma - f)^2$$

to yield

$$-\Delta v + v^\gamma \leq Df \cdot Du + f^2 - v^{\frac{\gamma-1}{2}} |Dv|.$$

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In this way, one obtains [Lions, '85]:

$$Du \in L^p \text{ for all } p, \quad f \in L^q, q > d$$

Theorem

Let $f \in C^1(\mathbb{T}^d)$, $\gamma > 1$,

$$q > d\frac{\gamma - 1}{\gamma} \quad \text{and } q > 2,$$

and $u \in C^3(\mathbb{T}^d)$ *be a classical periodic solution to H-J.*
Then, there exists $K = K(\|f\|_q, \|Du\|_1, \gamma, q, d) > 0$ *such that*

$$\|D^2 u\|_{L^q(\mathbb{T}^d)} + \|\,|Du|^\gamma\,\|_{L^q(\mathbb{T}^d)} \leq K.$$

Proof.

Via an (integral) Bernstein method: look at the equation satisfied by

$$w = g(|Du|^2) \sim |Du|$$

on its super-level sets, i.e. $\{w_k = (w - k)^+ \geq 0\}$:

$$-\Delta w_k + w^{2\gamma-1} \leq |Df| + \frac{f^2}{|Du|} - w^{\gamma-1}|Dw_k|.$$

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Multiply the equation w_k^β, $\beta > 1$, integrate, ... to get

$$Y_k = \iint w_k^{q\gamma}, \quad Y_k^{1-\frac{2}{d}} \leq Y_k + \varepsilon$$

control on Y_k.

(nonstandard approach in [Grenon-Murat-Porretta, Ann. Pisa '14])

the elliptic case: open problems

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- quasi-linear case? Δ replaced by Δ_p

the parabolic case

$$\partial_t u - \Delta u + |Du|^\gamma = f \quad \overset{?}{\Rightarrow} \quad \|\partial_t u\|_{L^q_{x,t}} + \|\Delta u\|_{L^q_{x,t}} + \|\, |Du|^\gamma \,\|_{L^q_{x,t}} \leq C(\, \|f\|_{L^q_{x,t}} \,)$$

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$$q \ge (d+2)\frac{\gamma - 1}{\gamma} = \bar{q}_{d+2,\gamma},$$

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- $\gamma = 2$: proof via Hopf-Cole (reduction to linear problem).

Taming the nonlinearity

By interpolation,

$$\|Du\|_{L^{\gamma q}} \lesssim \|D^2 u\|_{L^q}^{\theta} \|u\|_X{}^{1-\theta}.$$

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Writing $\quad \partial_t u - \Delta u = -|Du|^\gamma + f, \quad$ by (linear) max. regularity

$$\|\partial_t u\|_q + \|D^2 u\|_q \lesssim \|\,|Du|\,\|_{q\gamma}^{\gamma} + \|u_0\| + \|f\|_q$$

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$$\begin{aligned} \|\partial_t u\|_q + \|D^2 u\|_q &\lesssim \| \, |Du| \, \|_{q\gamma}^{\gamma} + \|u_0\| + \|f\|_q \\ &\lesssim \sup_t \|u(t)\|_X \cdot \ \|D^2 u\|_q^{\theta\gamma} + \|u_0\| + \|f\|_q \end{aligned}$$

We are done if $\theta\gamma < 1$ and $\sup_t \|u(t)\|_X < \infty$.

$$X = \begin{cases} L^p & \text{if } \gamma < 2, \\ C^{\alpha} & \text{if } \gamma \geq 2. \end{cases}$$

p, α depending on d, q.

Estimates in L^p / C^α

- L^p: [Magliocca '18]
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Unified method via duality à la [Evans '10]: linearize and consider the adjoint problem

$$-\partial_t \rho - \Delta \rho + \mathrm{div}(b\rho) = 0, \qquad b = -\gamma |Du|^{\gamma-2} Du$$

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Representation formula 1:

$$\int u(t)\rho(t) + \iint |b|^{\gamma'} \rho = \int u(0)\rho(0) + \iint f\rho$$

To estimate $\|u(t)\|_{L^p}$, let $\|\rho(t)\|_{L^{p'}} = 1$ and (try to) estimate $\rho \in L^{q'}$.

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Representation formula 2:

$$\iint \frac{u(x+h,t) - u(x,t)}{|h|^\alpha} \rho(t) \le \iint \frac{u(x+h,0) - u(x,0)}{|h|^\alpha} \rho(t) + \iint f(x,t) \frac{\rho(x-h,t) - \rho(x,t)}{|h|^\alpha}$$

To estimate $\|u(t)\|_{C^\alpha}$, let $\|\rho(t)\|_{L^1} = 1$ and (try to) estimate $\rho \in L_t^{q'} N_x^{\alpha,q'}$.

Regularity of the dual equation

Key fact: b in $\quad -\partial_t\rho - \Delta\rho + \mathrm{div}(b\rho) = 0 \quad$ satisfies

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Crucial Lemma:

$$\|D\rho\|_{q'} \lesssim \iint |b|^{\gamma'} \rho + \|\rho(t)\|_1$$

where

$$q' < \begin{cases} \frac{d+2}{d+1} & \text{if } \gamma \le 2 \\ 1 + \frac{\gamma'-1}{d+3-\gamma'} & \text{if } \gamma > 2\ . \end{cases}$$

obtained using linear max. regularity.

Theorem

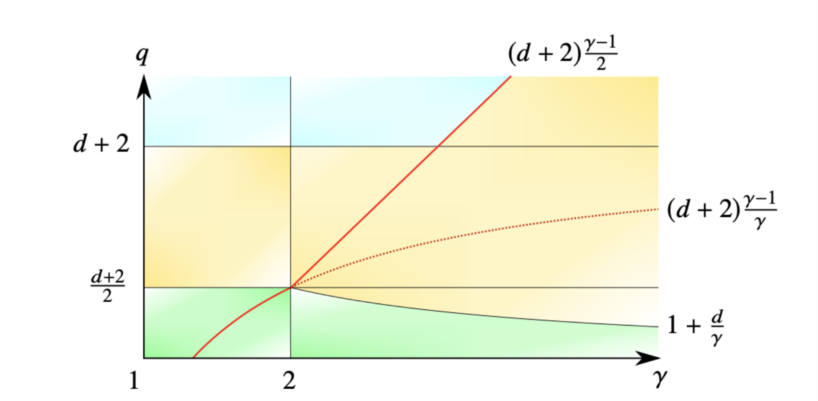
Let $u \in W_q^{2,1}(\mathbb{T}^d \times (0,T))$ be a strong solution to HJ and assume that for some $K > 0$

$$q > \begin{cases} (d+2)\frac{\gamma-1}{\gamma} = \bar{q}_{d+2,\gamma} & \text{if } \gamma < 2 \\ (d+2)\frac{\gamma-1}{2} > \bar{q}_{d+2,\gamma} & \text{if } \gamma \geq 2. \end{cases}$$

then, there exists a constant $C > 0$ depending on $\|f\|_{L^q(\mathbb{T}^d \times (0,T))}, \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}, q, d, T$ such that

$$\|u\|_{W_q^{2,1}(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \leq C.$$

The parabolic case: a summary



L^p estimates, Hölder estimates and Lipschitz estimates

The critical case

When $\gamma < 2$, we can prove max. regularity up to the threshold

$$q = (d+2)\frac{\gamma - 1}{\gamma},$$

using a stability argument. Indeed, by interpolation we have

$$\|D^2 u\|_{L^q} \lesssim \sup_t \|u(t)\|_{L^{d\frac{\gamma-1}{2-\gamma}}} \cdot \|D^2 u\|_{L^q} + \|u_0\| + \|f\|_q$$

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Approximate f with smooth $\tilde{f}$ and work on the equation involving $f - \tilde{f}$.
A byproduct of this procedure is that C in

$$\|u\|_{W^{2,1}_q(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \leq C$$

does not depend only on $\|f\|_{L^q(\mathbb{T}^d\times(0,T))}$, $\|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$!

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Thank you for your attention !