MA 571- Qualifying Exam- January 2021

Every problem is worth 14 points. Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use any theorem from Munkres' book but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't stated in Munkres and isn't obvious, be careful to give a clear explanation.

Problem 1 Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \overline{A} \subset \overline{U \cap A}$.

Problem 2: Denote by $[0, 1]$ the closed unit interval equipped with the subspace topology induced by the standard topology on $\mathbb{R}$. Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function then there is a point $x \in [0, 1]$ such that $f(x) = x$.

Problem 3: Let $X = \{1, 1/2, 1/3, ..., 1/n, ...\}$ and $Y = X \cup \{0\}$. Equip X and Y with the subspace topology induced by the standard topology on $\mathbb{R}$. Answer the following questions and prove your assertion directly from the definitions:
a) Is the space X compact?
b) Is the space Y compact?

Problem 4: Let $X = \mathbb{R}^2 - \{(0, 0)\}$ be the space obtained by removing the origin from the plane with the standard topology. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Answer the following questions and prove your assertions:
a) Are X and S^1 homeomorphic?
b) Are X and S^1 homotopy equivalent?

Problem 5: Suppose $p : X \to Y$ is a surjective, closed, continuous map between topological spaces such that for all $y \in Y$ the subspace $p^{-1}(y)$ is compact. Prove that X is compact.

Problem 6: Let X be the quotient space of the 2-sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ obtained by identifying $(0, 0, 1)$ and $(0, 0, -1)$ to a single point. Compute $\pi_1(X, x)$ for any $x \in X$ and justify your answer.

Problem 7: Let X be the quotient space obtained from an octagon P by pasting its edges together according to the labelling scheme $ada^{-1}bc^{-1}db^{-1}c$.
a) Calculate the first homology group $H_1(X)$.
b) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?