MA 571 Qualifying Exam. August 2019. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't stated in Munkres and isn't obvious, be careful to give a clear explanation.

1. Let $X = A \cup B$ and let C be a subset of $A \cap B$ which is closed in the subspace topology of A and also in the subspace topology of B. **Prove** that C is closed in the topology of X. (Hint: one way is to think about $\overline{C}$.)

2. Let X be a compact space and suppose we are given a nested sequence of subsets
$$C_1 \supset C_2 \supset \cdots$$
with all C_i closed. Let U be an open set containing $\cap C_i$.

   Prove that there is an i_0 with $C_{i_0} \subset U$.

3. Suppose that
   - X and Y are topological spaces and $f : X \to Y$ is continuous,
   - X is compact,
   - Y is Hausdorff, and
   - f is 1–1.

   Prove that X is homeomorphic to $f(X)$ (with the subspace topology).

4. Let A be the union of the x and y axes in $\mathbb{R}^2$. **Prove** that A is not homeomorphic to $\mathbb{R}$.

5. Suppose that
   - X and Y are topological spaces with continuous maps $Y \xrightarrow{f} X \xrightarrow{g} Y$,
   - $H : Y \times I \to Y$ is a homotopy from the identity map of Y to $g \circ f$, and
   - X is path connected.

   Prove that Y is path connected. (Hint: one way is to let $x_0 \in X$ and prove that there is a path from each y to $g(x_0)$.)

6. Let X be the quotient space obtained from an octagon P by pasting its edges together according to the labelling scheme $abab^{-1}cdc^{-1}d^{-1}$ (read the formula carefully!).

   i) Calculate $H_1(X)$. (You may use anything proved in Munkres, but be sure to be clear about what you're using.)

   ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?

7. Let $p : E \to B$ be a covering map and let $x \in B$. Suppose that U is a connected evenly covered neighborhood of x. **Prove** that for every connected component C of $p^{-1}(U)$, the map $p : C \to U$ is a homeomorphism.