# MA 571 Qualifying Exam. January 2015. Professor R. Kaufmann

## Instructions

There are 7 problems. Each problem is worth 14 points and you get two points for free.

**Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.**

**When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.**

## Problems

1. Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$ be continuous. Define $h : X \to \mathbb{R}$ by $h(x) = \min\{f(x), g(x)\}$. Use the pasting lemma to prove that $h$ is continuous. (You will not get full credit for any other method.)
2. Let $X$ and $Y$ be connected. Prove that $X \times Y$ is connected.
3. Recall that $g : X \to Y$ is called a proper map if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a map $f : X \to Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then $f$ is proper.
4. Let $A \subset X$; let $f : A \to Y$ be continuous; let $Y$ be Hausdorff. Show that if $f$ may be extended to a continuous function $g : \bar{A} \to Y$, then $g$ is uniquely determined by $f$.
5. Let $X$ be a Hausdorff space. Consider the equivalence relation on $X \times X$ generated by $(x, y) \sim (y, x)$ for all $x, y \in X$. Let $D$ be the diagonal, which is the image of $X$ under $\Delta : X \to X \times X; \Delta(x) = (x, x)$, and let $Z = (X \times X) \setminus D$, this is the subset consisting of points $(x, y)$ with $x \neq y$. Restrict $\sim$ to $Z$ and consider the quotient map $f : Z \to Z/\sim$.

   **Prove:** $f$ is a covering map.
6. Let $X$ be the space obtained by attaching two discs to $S^1$, where the first disc $D_1$ is attached via the map $\partial D_1 = S^1 \to S^1, z \to z^6$ and the second disc is attached by $g : \partial D_2 = S^1 \to S^1 : z \to z^4$. Compute $\pi_1(X)$ and $H_1(X)$. (Hint: use the special case of Seifert–van Kampen for adjoining 2–cells).
7. Let $S$ be the surface of genus $g$ with one hole, that is the $g$–fold torus $T_g$ with a region homeomorphic to an open disc removed. Prove that $\pi_1(S)$ is a free group and compute its rank.

S =