MA 571 Qualifying Exam. August 2015. Professor R. Kaufmann

Instructions

There are 8 problems. Each problem is worth 10 points

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

Problems

1. Let X_α be a family of topological spaces and $\prod_\alpha X_\alpha$ their product (with product topology). For each α, let A_α be a subset of X_α and $\overline{A_\alpha}$ its closure in X_α. Consider the subsets $\prod_\alpha \overline{A_\alpha}$ and $\prod_\alpha A_\alpha$ of $\prod_\alpha X_\alpha$. Let $\overline{\prod_\alpha A_\alpha}$ be the the closure of the second set in $\prod_\alpha X_\alpha$.

   Prove that
   $$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

2. Let X be a topological space and let $f, g : X \to [0, 1]$ be continuous functions.

   Suppose that X is connected and f is onto.

   Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

3. Consider the subspace of $\mathbb{R}^2$ consisting of two lines $S = \{(x, 0) : x \in \mathbb{R}\} \cup \{(x, 1) : x \in \mathbb{R}\} \subset \mathbb{R}^2$.

   (a) **Prove** that S is a locally compact Hausdorff space.

   (b) **Prove** that the one point compactification $\bar{S}$ of S is connected.

   For part (b), you may assume that part (a) holds.

4. Recall that $g : X \to Y$ is called a proper map if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a map $f : X \to Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then f is proper.

5. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

   Prove that the map
   $$e : X \times \mathcal{C}(X, Y) \to Y$$
   defined by the equation
   $$e(x, f) = f(x)$$
   is continuous.

6. Let $p : E \to B$ be a covering map. Let A be a connected space and let $a \in A$. Prove that if two continuous functions $\alpha, \beta : A \to E$ have the property that $\alpha(a) = \beta(a)$ and $p \circ \alpha = p \circ \beta$ then $\alpha = \beta$.

7. Consider the circle S^1 given as the quotient of the interval $[0,1] \subset \mathbb{R}$ by the equivalence relation $0 \sim 1$ and the Möbius band M, given as the quotient of the subspace $Sq = [0,1] \times [-1,1] \subset \mathbb{R}^2$ modulo the equivalence relation $(0,y) \sim (1,-y)$.

   Show that M is homotopy equivalent to S^1 as follows:

   (a) Show that the imbedding $i : [0,1] \to Sq, x \mapsto (x,0)$ induces an imbedding $\bar{\imath} : S^1 \to M$.

   (b) Let $I = i([0,1])$ be the image of i and $S = \bar{\imath}(S^1)$ the image of $\bar{\imath}$. Show that the retraction $r : Sq \to I, (x,y) \mapsto (x,0)$ induces a retraction $\bar{r} : M \to S$.

   (c) Give a deformation retraction $\bar{H} : Sq \times [0,1] \to Sq$ from Sq onto I, with $H(s,1) = r(s)$, which descends to a deformation retraction $\bar{H} : M \times [0,1] \to M$ from M to S with $\bar{H}(m,1) = \bar{r}(m)$.

   (d) Conclude the homotopy equivalence.

   (Be careful to state which properties of what map you are using in your proof and that you prove all of the required properties.)

8. Let F be the quotient space defined by pasting according to the *two word* labelling scheme $abcae^{-1}d^{-1} \quad bcde$.

   (a) Compute $H_1(F)$.

   (b) Is F a surface? If so, which one is it according to the classification theorem? (Justify your answers.)

   (c) Choose a point x_0 on F and compute $\pi_1(F, x_0)$.

   (You may, but do not have to, use the results of Problem 7).