MA 571 Qualifying Exam. January 2011. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let A be a subset of a topological space X and let B be a subset of A. **Prove** that $\bar{A} - \bar{B} \subset \overline{A - B}$.

2. Let G be a topological group (that is, a group with a topology for which the group operations are continuous) and let H be a subgroup of G. Suppose that G is connected, that H is a normal subgroup of G, and that the subspace topology on H is discrete. **Prove** that $gh = hg$ for every $g \in G, h \in H$.

3. Let X be the space with two points and the discrete topology. Let $Y = \prod_{n=1}^{\infty} X$, with the product topology. What are the connected components of Y? **Prove** that your answer is correct.

4. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. **Prove** that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subset N$.

5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is connected. **Prove** that Y is connected.

6. Let $p : E \to B$ be a covering map. Let $e_0 \in E$ and $b_0 \in B$ with $p(e_0) = b_0$.

   Let Y be simply connected (in particular, Y is path-connected). Let $y_0 \in Y$.

   Let $f : Y \to B$ be continuous, with $f(y_0) = b_0$.

   Prove that the following function $g : Y \to E$ is well-defined: given $y \in Y$, choose a path γ from y_0 to y; let β be the lift of $f \circ \gamma$ to E starting at e_0; now define $g(y) = \beta(1)$.

   You may use the fact (without having to prove it) that the lift of a path homotopy is again a path homotopy.

7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

   $$\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}.$$

   Let x_0 be the point $(0, 0, 1)$ of S^2.

   Use the Seifert-van Kampen theorem to **prove** that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert-van Kampen theorem given in Munkres's book. You will **not** get credit for any other method.