MA 571 Qualifying Exam. January 1998.

Each problem is worth 14 points and you get two points for free.

1. (a) Let X be a metric space and let Y be a subset of X. Suppose that the induced metric on Y is the discrete metric (i.e., the distance between any two points of Y is 1 unless the points are the same). Prove that Y is closed as a subset of X.

   (b) Give an example of a topological space X with a subset Y so that the induced topology on Y is the discrete topology but Y is not closed as a subset of X.

2. Let X be a topological space and let A be a dense subset of X. Let Y be a Hausdorff space, and let $g, h : X \to Y$ be continuous functions which agree on A. Prove that $g = h$.

3. Let X and Y be connected. Prove that $X \times Y$ is connected.

4. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

5. Prove that every covering map is an open map (recall that an open map is a function that takes open sets to open sets).

6. Let $p : E \to B$ be a covering map with E path-connected. Let $p(e_0) = b_0$.

   (a) Give the definition of the standard map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ constructed in Munkres (you do NOT have to prove that this is well-defined).

   (b) Suppose that α and β are two elements of $\pi_1(B, b_0)$ with $\phi(\alpha) = \phi(\beta)$. Prove that there is an element γ of $\pi_1(E, e_0)$ with $\beta = p_*(\gamma) \cdot \alpha$.

7. Let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Recall that h induces a homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$, which will be denoted $(h_{x_0})_*$, and also a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(Y, y_1)$, which will be denoted $(h_{x_1})_*$. Suppose that X is path-connected. Prove that there are isomorphisms

$$\phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

and

$$\psi : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$$

so that

$$\psi \circ (h_{x_0})_* = (h_{x_1})_* \circ \phi$$