Qualifying Examination
MA 571
August 1997

Let $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$, the standard orthonormal basis for $\mathbb{R}^2$.

1. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ given by $\vec{v} \sim (\vec{v} + m\vec{i} + n\vec{j})$ for m and n integers. Then the identification map $q : \mathbb{R}^2 \to \mathbb{R}^2/\sim = T$ is a covering map of the torus T. Let $\sigma_{\vec{v}} : \mathbb{R} \to \mathbb{R}^2 \xrightarrow{q} T$ be a map given by $\sigma_{\vec{v}}(t) = q(t\vec{v})$.
   a) Show that the image $\sigma_{\vec{v}}(\mathbb{R})$ is dense in T if and only if $\vec{v} := a\vec{i} + b\vec{j}$ is such that a/b is irrational.
   b) Show that $\sigma_{\vec{v}}(\mathbb{R})$ is compact if and only if a/b is rational.

2. Let A be a subset of a space X.
   a) $\overline{\overset{\circ}{A}} \subset \overset{\circ}{\overline{A}}$ Prove or give counter example.
   b) $\overline{\overset{\circ}{A}} \supset \overset{\circ}{\overline{A}}$ Prove or give counter example.

3. Give examples of:
   a) Two different metrics on the same set X which give non homeomorphic topological spaces.
   b) Two different metrics on the same set Y which give homeomorphic topological spaces.

4. Give examples of:
   a) A map which is both open and closed.
   b) A map which is neither open nor closed.
   c) A map which is open but not closed.
   d) A map which is closed but not open.

5. The fundamental group $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\triangle : S^1 \to S^1 \times S^1 : s \mapsto (s,s)$ be the diagonal map. Can $\triangle$ be lifted to a map $\tilde{\triangle} : S^1 \to S$ where S is the covering space of $S^1 \times S^1$ corresponding to the subgroup $\mathbb{Z} \oplus 0$ in $\pi_1(S^1 \times S^1)$? Why?
6. Show that the diagonal of $X \times X$ is closed if and only if X is Hausdorff.
7. A <u>fibre bundle</u> $p : E \to B$ is a map so that $p^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times F$ for some covering $\{U_\alpha\}$ of B by closed neighborhoods, where F is a space homeomorphic to every fibre $p^{-1}(b)$. Assume E and B are Hausdorff. Then prove that E is compact if and only if F and B are compact.