MA 562 Qualifying Exam
January 2022

Solve all six problems. Justify your answers. Each problem is worth 10 points.

1. Let M be a manifold of dimension n and let a and b be real numbers with $a < b$. Let $f\colon M \to (a,b)$ and $g\colon (a,b) \to M$ be C^∞ functions. If $f(g(t)) = t$ for every $t \in (a,b)$, does it follow that $n = 1$? If $g(f(x)) = x$ for every $x \in M$ does it follow that $n = 1$? In each case, give a proof or a counterexample.

2. For each $a \in \mathbb{R}$, let let $f_a\colon \mathbb{R} \to \mathbb{R}^2$ be given by
$$f_a(t) = (\sin(a \arctan t), \sin(2a \arctan t)).$$
   (a) For which values of a is f_a an immersion?

   (b) For which values of a is f_a injective?

   (c) For which values of a is f_a an embedding?

3. Let M and N be manifolds, and let Γ be a subset of N. Let $\mathcal{I}$ be the set of injective immersions $f\colon M \to N$ such that $f(M) = \Gamma$. For f and g in $\mathcal{I}$, define the relation $f \sim g$ to mean that there is a diffeomorphism h of M such that $f \circ h = g$.
   (a) Prove that $\sim$ is an equivalence relation.

   (b) Prove that if $f \in \mathcal{I}$ and $g \in \mathcal{I}$ are embeddings, then $f \sim g$.

   (c) In the case that $M = \mathbb{R}$ and $N = \mathbb{R}^2$, give an example of $\Gamma \subset N$ and $f \in \mathcal{I}$ and $g \in \mathcal{I}$ such that $f \not\sim g$.

4. Let N be a closed regular submanifold of a manifold M. Let $f \in C^\infty(N)$.
   (a) Show there is a function $g \in C^\infty(M)$ such that $f(p) = g(p)$ for all $p \in N$.

   (b) Suppose additionally that f is bounded. Show that there is a function $h \in C^\infty(M)$ such that $f(p) = h(p)$ for all $p \in N$ and such that $|h(p)| \le \sup |f|$ for all $p \in M \setminus N$.

   (c) Under what necessary and sufficient additional condition on f can we replace the '$\le$' in part (b) by '$<$'?

5. Let A be an antisymmetric 3×3 matrix, and consider the vector field on $\mathbb{R}^3$ given by $V(x) = Ax$.
   (a) Prove that $V(x)$ is tangent to all spheres in $\mathbb{R}^3$ centered at the origin.

   (b) For any $w \in \mathbb{R}^3$ such that $Aw = 0$, prove that $V(x)$ is tangent to all spheres in $\mathbb{R}^3$ centered at w.

   (c) Let $\theta(t,p)$ be the one-parameter group on the unit sphere S^2 corresponding to $V(x)$. Prove that for every $p \in S^2$, there is a circle C_p containing the orbit $\{\theta(t,p)\colon t \in \mathbb{R}\}$.

6. Let $f\colon (0,\infty) \to \mathbb{R}$ be a C^∞ function and define a differential 1-form on $\mathbb{R}^n \setminus \{0\}$ by

$$\omega = f(|x|^2) \sum_{j=1}^{n} x^j \, dx^j .$$

(a) Let $G\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be given by $G(y) = \exp(-|y|^2)y$. Compute $G^*\omega$.

(b) Compute $d\omega$.

(c) If f is the natural logarithm function, do there exist functions $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $dh = \omega$? If yes, give an explicit formula for all of them. If no, prove it.