**Math 562: Winter 2018 Qualifying Exam (McReynolds)**

**PUID Number:**______________________________________

Work **<u>four out of five</u>** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded.

**Problem 1.** [15 points] Let $M$ be a smooth connected manifold, $p,q \in M$, $v \in T_p(M)$, and $w \in T_q(M)$. Prove that there exists a diffeomorphism $f\colon M \to M$ with $f(p) = q$ and $df_p(v) = w$.

**Problem 2.** [15 points] Let $M_1, M_2, N$ be smooth manifolds and $V_1, V_2, W$ be finite dimensional vector spaces.

(a) Let $L_j\colon V_j \to W$ be linear maps for $j = 1,2$. Prove that if $L_1$ is onto, then

$$\mathrm{Image}(L_1) \oplus \{0\} + \{0\} \oplus \mathrm{Image}(L_2) + \Delta_W = W \times W$$

where $\Delta_W$ is the diagonal in $W \times W$.

(b) Let $f_j\colon M_j \to N$ be smooth maps for $j = 1,2$. Prove that if $f_1$ is a submersion, then

$$Z = \{(p_1, p_2) \in M_1 \times M_2 \ : \ f_1(p_1) = f_2(p_2)\}$$

is a submanifold.

**Problem 3.** [15 points] Let $f\colon \mathbf{R}^5 \to \mathbf{R}^3$ be a smooth map. Prove that there exists a sphere $S \subset \mathbf{R}^3$ centered at the origin such that $f^{-1}(S)$ is a smooth submanifold of $\mathbf{R}^5$.

**Problem 4.** [15 points] Let $M$ be a compact, orientable $n$–manifold with boundary $\partial M$ and let $i\colon \partial M \to M$ be the inclusion map. For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{n-k-1}(M)$ such that $i^*(\beta) = 0$, prove that

$$\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta.$$

**Problem 5.** [15 points] Let

$$\omega = z dx \wedge dy + (z^2 e^{xz} \ln(x^2+1) - y) dx \wedge dz + (x + y^2 z^3 \cos(y^2+z^2)) dy \wedge dz$$

be a smooth 2–form on $\mathbf{R}^3$.

(a) Is $\omega$ closed? Is $\omega$ exact? Prove you answers.

(b) Let $S$ be the surface given by the union of the two sets

$$C = \left\{(x,y,z) \ : \ x^2+y^2 = 1, \ z \in [-1,1]\right\}$$
$$H = \left\{(x,y,z) \ : \ x^2+y^2+(z-1)^2 = 1, \ z \geq 1\right\}.$$

Compute

$$\int_S \omega.$$

Please specify which orientation you are using.