Math 562: Summer 2019 Qualifying Exam (McReynolds)

PUID Number:______________________________________

Work the following problems. The time limit is two hours.

Problem 1. [20 points]

Let M,N be smooth manifolds and let $f\colon M \to N$ be a smooth function. Define the following:

(a) For f to be an immersion at $p \in M$.

(a) For f to be a local diffeomorphism.

(c) For $q \in N$ to be a regular value.

(d) For f to be proper.

Problem 2. [20 points]

State whether the following statement is true or false. You do not need to justify your answer.

(a) If M is a smooth manifold and $C \subset M$ is a closed subset, then C is a submanifold of M.

(b) If $f\colon M \to N$ is an immersion, then $\dim(M) \leq \dim(N)$.

(c) If $f\colon M$ is a smooth bijective function, then f is a diffeomorphism.

(d) If $\alpha \in \Omega^k(M)$ is a closed k–form, then $d\alpha = 0$.

Problem 3. [20 points]

Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x,y) = (x^2+y, y^2-x)$. Compute the following:

(a) df_p for $p = (x_0, y_0)$.

(b) $f^*(dx)$.

(b) $f^*(dy)$.

(c) $f^*(dx \wedge dy)$.

Problem 4. [10 points]

Prove that the set

$$\left\{(x,y,z) \in \mathbf{R}^3 \ : \ x^2+2xz-2yz-y^2 = 0,\ 2x-y+z = 3\right\}$$

is a submanifold of $\mathbf{R}^3$ in the subspace topology.

Problem 5. [20 points]

Let

$$M = \left\{(x,y,z)\ :\ x^2+y^2+1 = z^2,\ z>0\right\}$$

and

$$S = \left\{(x,y,z)\in M\ :\ x^2+y^2\leq 3\right\}.$$

(a) Prove that S is a manifold with boundary in the subspace topology.

(b) Compute

$$\int_S \omega$$

where

$$\omega = dx\wedge dy + (4-2z)dx\wedge dz + (z-2)dy\wedge dz.$$

Problem 6. [10 points]

Let X be a compact, oriented n–manifold. Prove that for any smooth differential $(n-1)$–form $\omega \in \Omega^{n-1}(X)$, there exists a point $x\in X$ where $d\omega = 0$.

Problem 7. [20 points]

Let $f\colon X\to Y$ be a smooth map between smooth compact, orientable n–manifolds.

(a) Prove that if there exists a smooth differential n–form $\omega\in\Omega^n(Y)$ such that

$$\int_X f^*(\omega)\neq 0,$$

then f is surjective.

(b) Is the converse true? Justify your answer.