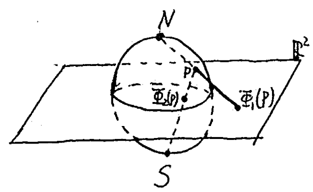
MA 562 Qualify Exam January 2018

1. Let $\mathbb{S}^2 = \{(x, y, z); x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere. Denote the north pole by $N = (0, 0, 1)$ and the south pole by $S = (0, 0, -1)$. Denote $U_1 = \mathbb{S}^2 \setminus \{N\}$ and $U_2 = \mathbb{S}^2 \setminus \{S\}$. For any $p \in S^2 \setminus \{N\}$, denote by $\overline{pN}$ the line in $\mathbb{R}^3$ passing through p and N. Use similar notation for $\overline{pS}$.



The stereographic projection $\Phi_1 : U_1 = \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2 = \{(x, y, 0); (x, y) \in \mathbb{R}^2\}$ is defined as:

$$\Phi_1(p) = \overline{pN} \cap \{z = 0\}, \quad \text{for any } p \in U_1.$$

Similarly $\Phi_2 : U_2 := \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$ is defined as

$$\Phi_2(p) = \overline{pS} \cap \{z = 0\}, \quad \text{for any } p \in U_2.$$

Prove that $(U_1 = \mathbb{S}^2 \setminus \{N\}, \Phi_1)$, $(U_2 = \mathbb{S}^2 \setminus \{S\}, \Phi_2)$ define a differentiable structure on $\mathbb{S}^2$. What is the transition function between these two coordinate charts?

2. Is there a smooth vector field on the 2-dimensional torus $S^1 \times S^1$ with a single zero point? What is the index of that zero point if such a vector field exists?

3. Denote by $M = \text{Mat}_{3\times 2}(\mathbb{R}) \cong \mathbb{R}^6$ the set of 3×2 real matrices. A^T denotes the transpose of A. Prove that the set $N = \{A \in \text{Mat}_{3\times 2}(\mathbb{R}); A^T A = I_2\}$ is a smooth manifold. What is its dimension?

4. Prove the following result:

   Let M be a smooth manifold. A smooth 1-form ω on M is an exact form if and only if, for any closed piece-wisely smooth curve C on M, $\int_C \omega = 0$.

5. Let $i : M \to N$ be an immersed submanifold and X be a smooth vector field on M.

   (a) If M is a smooth embedded submanifold, prove that there exists a smooth vector field $\tilde{X}$ on N such that $\tilde{X}_p = i_*(X_p)$ for any $p \in M$.

   (b) Is the above statement true if i is only an injective immersion but not an embedding?

6. Consider the distribution Δ on $\mathbb{R}^3$ defined as the kernel of α where:

$$\alpha = y dx - x dy + dz.$$

   In other words, $\Delta_p = \text{Ker}(\alpha_p)$ for any $p \in \mathbb{R}^3$. Is the distribution Δ integrable?