Qualifying Examination
January 2011
Math 562 – Professor Donnelly

1. Let M be a nonempty smooth manifold of dimension $n \geq 1$. Show that $C^\infty(M)$ is infinite dimensional.

2. Prove that the elements $v_1, v_2, \ldots, v_r$ of the vector space V are linearly independent if and only if $v_1 \wedge v_2 \wedge \cdots \wedge v_r \neq 0$.

3. Show that the Lie derivative on tensor fields satisfies $L_X L_Y - L_Y L_X = L_{L_X Y}$.

4. Deduce the divergence theorem of advanced calculus from the general Stokes' theorem on manifolds.

5. Give an example of a 3–tensor that cannot be written as the sum of an alternating tensor and a symmetric tensor.

6. Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Prove that G is abelian if and only if $\mathfrak{g}$ is abelian.

7. Give an example of a complete noncompact Riemannian manifold for which there is no geodesic, defined on $(-\infty, \infty)$, which minimizes the distance between any two points in its image.

8. Show that the distribution on R^4 spanned by $\partial_y + x\partial_z$ and $\partial_x + y\partial_w$ has no 2–dimensional integral submanifolds.

9. Prove that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not e^A for any $A \in \mathfrak{g}\ell(2, R)$.

10. Let Φ be a differentiable mapping from a manifold M onto a manifold N. A vector field X is projectable if there exists a vector field Y on N such that $d\Phi(X) = Y$. If $d\Phi_p$ is surjective, for all $p \in M$, and $d\Phi_p(X_p) = d\Phi_q(X_q)$, whenever $\Phi(p) = \Phi(q)$, show that X is projectable.