# MA562 Qualifying Exam 1/2/07

*Print your name:*________________________________________

**1.** The *hyperbolic space* $\mathbf{H}^n$ is the unit disc $\{(x_1,\ldots,x_n)\,|\,\sum_i x_i^2<1\}$ with the *hyperbolic metric* $g_H=\left(\frac{2}{1-\sum_i x_i^2}\right)^2 g_E$, where $g_E$ is the Euclidean metric. Consider the parametrized surface $S=F(D^2)\subset\mathbf{H}^3$, where $F(u,v):=(u,v-u,v)$ and $D^2=\{(u,v)\,|\,u^2+v^2<1/4\}$. Let $\Omega$ be the volume form on $S$ associated to the metric induced from the hyperbolic metric, and the orientation determined by the parametrization.

(a) Express $F^*\Omega$ in $du,dv$.

(b) Use the Stokes' theorem to express the area of $S$ as a line integral over $\partial D$.

**2.** Let $SL(2,\mathbb{R})\subset GL(2,\mathbb{R})$ be the Lie group of $2\times 2$ real matrices of determinant 1. Let $x_{ij}$ denote the *global* coordinate function on $GL(2,\mathbb{R})$ which sends a matrix to its $ij$-th entry. Let $e$ denote the identity.

(a) Let $v$ be a left invariant vector field on $GL(2,\mathbb{R})$. Then $v(g)=\sum_{ij}f_{ij}(g)\frac{\partial}{\partial x_{ij}}$, where $f_{ij}$ is a function on $GL(2,\mathbb{R})$. Express the matrix $(f_{ij}(g))$ in terms of the matrix $(f_{ij}(e))$ for a general $g\in GL(2,\mathbb{R})$.

(b) It is known that by sending $v$ to the matrix $(f_{ij}(e))$ above, the Lie algebra of $GL(2,\mathbb{R})$ can be identified with the Lie algebra of $2\times 2$ matrices and commutators, and hence the Lie algebra of $SL(2,\mathbb{R})$ is identified with a Lie subalgebra of the latter. Show that $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ is an element in this Lie subalgebra, and find the 1-dimensional subgroup of $SL(2,\mathbb{R})$ which is an integral manifold of the corresponding invariant vector field.

**3.** Let $\mathbb{T}^2=(\mathbb{R}/\mathbb{Z})^2$. Show that $\Omega^*(\mathbb{T}^2)$ is a free module over $C^\infty(\mathbb{T}^2)$, and compute the de Rham cohomology groups of $\mathbb{T}^2$.

**4.** Let $M$ be a manifold of dimension $m$ and $\alpha$ be a nowhere vanishing 1-form on $M$. Let $\xi_\alpha$ denote the $(m-1)$-plane distribution $\ker\alpha$, regarding $\alpha(x)$ as a linear map from $T_xM$ to $\mathbb{R}$.

(a) Show that $\xi_\alpha$ is integrable iff $\alpha\wedge d\alpha=0$.

(b) Suppose $m=2n+1$ is odd. Show that the following two conditions are equivalent:
(i) The bilinear form $d\alpha$ on $T_xM\times T_xM$ is nondegenerate when restricted on $\xi_\alpha(x)\times\xi_\alpha(x)$ for every $x\in M$;
(ii) $\alpha\wedge(d\alpha)^n$ is nowhere vanishing.
(Hint: Assume the following fact from linear algebra: If $V$ is a finite dimensional vector space and $w$ is a nondegenerate skew-symmetric bilinear form on $V\times V$, then there is a basis $\{u_1,\ldots,u_k,v_1,\ldots,v_k\}$ of $V$, such that $w(u_i,u_j)=w(v_i,v_j)=0$, $w(u_i,v_j)=\delta_{ij}\ \forall i,j$.)

(c) A 1-form $\alpha$ satisfying the conditions in (b) above is called a *contact form*. Let $N$ be a Riemannian manifold. Show that the 1-form $\alpha_{can}$ on $T^*N$ defined below restricts to a contact form on the unit cotangent bundle of $N$: Let $V$ be a tangent vector at $(x,p)\in T^*N$, where $x\in N$ and $p\in T_x^*N$, and let $\pi:T^*N\to N$ be the projection map. Then the pairing $\langle\alpha_{can}(x,p),V\rangle:=\langle p,\pi_*V\rangle$.

**5.** Let $M$ be a compact connected Riemannian manifold, and $f : M \to S^1 = \mathbb{R}/\mathbb{Z}$ be a submersion. For $t \in S^1$, denote the *level hypersurfaces* $f^{-1}(t)$ by $N_t$.

(a) Show that $N_t$ are embedded submanifolds, and they are orientable iff $M$ is.

(b) Let $v_f$ denote the vector field dual to the 1-form $df$. Show that $v_f$ is nowhere vanishing and complete, and that $f$ is surjective.

(c) Show that $M$ can not be simply connected.