Qualifying Examination
January 2006
Math 562 – Professor Donnelly

1. If M is a compact manifold, show that every continuous map $M \to S^p$ can be uniformly approximated by a smooth map.

2. If $\dim(M) < p$, show that every continuous map $M \to S^p$ is homotopic to a constant.

3. Consider $f(x,y) = \left(\dfrac{x^2 - y^2}{x^2 + y^2}, \dfrac{xy}{x^2 + y^2}\right)$. Does this map from $R^2 - (0,0)$ to R^2 have a local inverse near $(x,y) = (0,1)$?

4. If $F : G \to H$ is a Lie group homomorphism, show that the kernel of $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is the Lie algebra of the kernel of F.

5. Show that every closed 1–form on the open shell $1 < \|x\| < 2$ in R^3 is exact. Find a 2–form on the same shell which is closed but not exact.

6. Determine the flow associated to the vector field $V = x_2 \dfrac{\partial}{\partial x_1} + x_3 \dfrac{\partial}{\partial x_2} + x_1 \dfrac{\partial}{\partial x_3}$ in R^3.

7. Is it possible to solve $xy^2 + xzu + yv^2 = 3$, $u^3yz + 2xv - u^2v^2 = 2$ for $u(x,y,z)$, $v(x,y,z)$ near $(x,y,z) = (1,1,1), (u,v) = (1,1)$?

8. Show that $TS^n \times R$ is diffeomorphic to $S^n \times R^{n+1}$.

9. Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Define $F : M \to M \times N$ by $F(x) = (x, f(x))$. Show that for every smooth vector field V on M, there is a smooth vector field on $M \times N$ that is F–related to V.

10. Show that the matrix exponential satisfies $\det(e^A) = e^{TrA}$.