QUALIFYING EXAMINATION

JANUARY 2000

MATH 562 - Prof. Penney

Notation: Unless stated otherwise, all objects belong to the C^∞ category. Thus, all manifolds, vector fields, functions, etc. are assumed to be C^∞ unless stated to the contrary. "Submanifold" means "immersed submanifold."

(1) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = x^3 + xy + y^3$.

(a) Show that $M = f^{-1}(1)$ is an imbedded submanifold of $\mathbf{R}^2$.

(b) Show that $M = f^{-1}(0)$ is not a submanifold of $\mathbf{R}^2$. *Hint:* Prove that otherwise there would be a pair of C^∞ functions x and y satisfying $f(x, y) = 0$ where $x(0) = y(0) = 0$ and $(x'(0), y'(0)) \neq (0, 0)$. Then consider the existence of $\lim_{t\to 0} y(t)/x(t)$ and $\lim_{t\to 0} x(t)/y(t)$.

(2) Let M be a submanifold of N.

(a) Show that if X and Y are vector fields on N that are tangent to M at each point of M, then $[X, Y]$ is also tangent to M at each point of M.

(b) Which of the following pairs of vector fields on $\mathbf{R}^3$ is tangent to a two dimensional submanifold of $\mathbf{R}^3$ containing $(1, 1, 1)$ and which is not? Prove you answers

(a) $$X_1 = e^y \frac{\partial}{\partial y} + x e^y \frac{\partial}{\partial z} \qquad X_2 = (1 + x^2)\frac{\partial}{\partial x} - y(1 + x^2)\frac{\partial}{\partial z}$$

(b) $$X_1 = x^2 y \frac{\partial}{\partial x} - \frac{\partial}{\partial z} \qquad X_2 = x y^2 \frac{\partial}{\partial y} - \frac{\partial}{\partial z}$$

(3) True or false. Justify your answer by either quoting a relevant theorem or providing a counter example.

(a) All vector fields on the two sphere S^2 are complete.

(b) All vector fields on $\mathbf{R}$ are complete.

(4) Let A and B be disjoint closed subsets of a manifold M. Prove that there is a C^∞ function f on M which is identically 1 on A and identically 0 on B.

(5) A C^∞ function f on $\mathbf{R}^n$ is said to be harmonic if $\Delta f = 0$ where

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Show that f is harmonic if and only if for all $n-1$ spheres S with arbitrary center and radius,

$$0 = \int_S \sum_i (-1)^i \frac{\partial f}{\partial x_i} dx_1 \wedge dx_2 \ldots \wedge d\hat{x}_i \wedge \ldots dx_n$$

where "$d\hat{x}_i$" indicates that the i^{th} wedge product is omitted. *Hint:* If $\Delta f(x_o) \neq 0$, then there is a closed ball containing x_o on which $\Delta f \neq 0$.

(6) Prove that two dimensional projective space is a homogeneous space for $SO(3)$, the group of 3×3 orhtogonal matrices of determinant 1. Describe the Lie algebra of the isotropy subgroup K and use it to prove that K is one-dimensional.

(7) Let N be an n-dimensional oriented Riemannian manifold and let $M \subset N$ be an $n-1$-dimensional oriented submanifold given the induced Riemannian structure. Let Ω_N and Ω_M be the corresponding Riemannian volume forms. For each $m \in M$, let $N(m) \in T_m(N)$ be defined by (i) $||N(m)|| = 1$, (ii) $N(m)$ is orthogonal to $i^*(T_m(M))$, (iii) if $\{X_1, X_2, \ldots, X_{n-1}\}$ is a positively oriented basis for $T_m(M)$ then $\{N(m), i^*(X_1), i^*(X_2), \ldots, i^*(X_{n-1})\}$ is a positively oriented basis for $T_m(N)$ where $i : M \to N$ is the injection mapping.

(a) Describe a C^∞ structure on the tangent bundle $T(N)$ of N. (You need not prove that your C^∞ structure really is a C^∞ structure.) Prove that $\phi : m \to N(m)$ is differentiable relative to this structure. *Hint:* Use a local orthonormal frame for M.

(b) Prove that for $X_1, X_2, \ldots, X_{n-1}$ in $T_m(M)$,

$$\Omega_M(X_1, X_2, \ldots, X_{n-1}) = \Omega_N(N(m), i^*(X_1), i^*(X_2), \ldots, i^*(X_{n-1}))$$

(c) Let $M \subset \mathbf{R}^n$ be the graph of a C^∞ function $f : \mathbf{R}^{n-1} \to \mathbf{R}$ and let coordinates on M be defined by projection onto the first $n-1$ coordinates. Use part (b) to show that in these coordinates

$$\Omega_M = \sqrt{1 + f_1^2 + \cdots + f_{n-1}^2}\, dx_1 \wedge dx_2 \ldots \wedge dx_{n-1}$$

where $f_i = \dfrac{\partial f}{\partial x_i}$.