Math 55400 **Qualifying Exam** **January 14, 2021** **W. Heinzer**

PUID: ______________________________

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	18	
3	22	
4	18	
5	18	
6	18	
7	20	
8	20	
9	20	
10	18	
11	18	
12	10	
Total	200	

1. (12 pts) Let V be a finite-dimensional vector space over a field F and let W_1 and W_2 be subspaces of V. Prove that

$$\dim W_1 \quad + \quad \dim W_2 \quad = \quad \dim(W_1 \cap W_2) \quad + \quad \dim(W_1 + W_2).$$

2. (6 pts) Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for each vector α in the subspace W.

3. Let V and W be finite-dimensional vector spaces over a field F, and let $T : V \to W$ be a linear transformation.

(a) (2 pts) Define the rank of T.

(b) (2 pts) Define the nullity of T.

(c) (10 pts) State and prove a theorem involving the rank of T and the nullity of T.

4. (8 pts) Let F be an infinite field and let $g \in F[x]$ be a monic polynomial of degree $n > 0$.

(a) Describe the ideals in $F[x]$ that contain g.

(b) Are there finitely many or infinitely many ideals in $F[x]$ that contain g?

5. (12 pts) Let F be a field, let S be a set, and let $\mathcal{F}(S, F)$ be the set of all functions from S to F.

(a) As in Chapter 2 of Hoffman and Kunze, define vector addition and scalar multiplication on the set $\mathcal{F}(S, F)$ so that $\mathcal{F}(S, F)$ is a vector space over the field F.

(b) If S is a finite set with n elements what is the dimension of the vector space $\mathcal{F}(S, F)$? Justify your answer.

6. (6 pts) State true or false and justify your answer: If V is a finite-dimensional vector space and W_1 and W_2 are subspaces of V such that $V = W_1 \oplus W_2$, then for any subspace W of V we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

7. Define the following terms as in Hoffman and Kunze.

(a) (4 pts) $\mathfrak{A}$ is a *linear algebra over the field* F.

(b) (4 pts) The vector space V of *polynomial functions over a field* F.

(c) (4 pts) The vector space $F[x]$ of *polynomials over a field* F.

8. (6 pts) For what fields F is the vector space of polynomial functions over F isomorphic to the vector space of polynomials over F? Justify your answer.

9. Let D be a principal ideal domain and let M be a finitely generated D-module.

(a) (3 pts.) What does it mean for a subset $S = \{z_1, \ldots, z_n\}$ of M to be a *generating set* for M?

(b) (3 pts.) What does it mean for a subset $S = \{z_1, \ldots, z_n\}$ of M to be a *basis* for M.

(c) (6 pts.) What does it mean for a matrix $A \in D^{m \times n}$ to be a *relation matrix* for M? How is a relation matrix for M constructed?

(d) (6 pts.) State true or false and justify your answer with either a proof or a counterexample: Every nonzero finitely generated D-module has a basis.

10. (20 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space V, and let $\mathcal{F}$ be the vector space of linear operators $U : V \to V$ that commute with T.

(a) Prove that $\dim \mathcal{F} \geq n$.

(b) Prove that T has a cyclic vector if and only if every $U \in \mathcal{F}$ is a polynomial in T.

11. (20 pts) Let V be an abelian group generated by elements a, b, c. Assume the following relations hold: $2a = 4b, 2b = 4c, 2c = 4a$, and these three relations generate all the relations on a, b, c.

(a) Write down a relation matrix for V.

(b) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by x, y, z, and express your generators x, y, z in terms of a, b, c.

(c) What is the order of V?

(d) What is the order of the element a?

12. (10 pts) Let V be a finite-dimensional vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

13. (10 pts) Let V be a finite-dimensional vector space over an infinite field F and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in V. Prove that there exists a linear functional f on V such that $f(\alpha_i) \neq 0$ for each i with $1 \leq i \leq m$.

14. (18 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space over a field F. Let $c_1, \ldots, c_k$ be distinct elements in F and let $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ be the minimal polynomial of T. Let $W_i = \{v \in V \mid (T - c_i I)^{r_i}(v) = 0\}$.

(a) Describe linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) Describe how to obtain linear operators D and N such that $T = D + N$, where D is diagonalizable, N is nilpotent and D and N are polynomials in T.

(c) If $T = D' + N'$, where D' is diagonalizable and N' is nilpotent and $D'N' = N'D'$, prove that $D = D'$ and $N = N'$.

15. (18 pts) Let notation be as in the previous problem and let $f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ be the characteristic polynomial for T. Thus $n = d_1 + \cdots + d_k$ and $1 \leq r_i \leq d_i$ for each i.

(a) If $r_i + 1 = d_i$ for each $i \in \{1, \ldots, k\}$, describe the Jordan form for T.

(b) If $r_i + 2 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible for T?

(c) If $r_i + 3 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible for T?

16. (10 pts.) Let V be an n-dimensional vector space over a field F.

(a) True or false: Every monic polynomial in $F[x]$ of degree n is the characteristic polynomial of some linear operator on V. Justify your answer.

(b) True or false: Every monic polynomial in $F[x]$ of degree n is the minimal polynomial of some linear operator on V. Justify your answer.