554 QUALIFYING EXAM, JAN 8 2019

Attempt all questions. Time 2 hrs.

1. (5 + 5 pts) Let A be a commutative ring with unit, E, F be two A-modules and $u : E \to F$ a linear mapping.
   (a) Show that the mapping $(x, y) \mapsto (x, y - u(x))$ of the product module $E \times F$ to itself is an automorphism of $E \times F$.
   (b) Deduce that if there exists a linear mapping $v : F \to E$, and an $a \in E$, such that $v(u(a)) = a$, there exists an automorphism of w of $E \times F$ such that $w(a, 0) = (0, u(a))$.
2. (10 pts) Let k be a field, E a finite dimensional k-vector space, and $u, v \in \mathrm{End}(E)$. Prove that
$$\dim \ker(u \circ v) \leq \dim \ker(u) + \dim \ker(v).$$
3. (5 + 10 pts)
   (a) Define the rank of a linear mapping between two vector spaces.
   (b) Let E, F, G, H be four finite dimensional vector spaces over a field k and $u : E \longrightarrow F, v : F \longrightarrow G, w : G \longrightarrow H$ be linear mappings. Prove that
$$\mathrm{rk}(v \circ u) + \mathrm{rk}(w \circ v) \leq \mathrm{rk}(v) + \mathrm{rk}(w \circ v \circ u).$$
4. (5 + 5 + 5 pts) Let k be a field, E a finite dimensional k-vector space, and let F be a set of endomorphisms of E such that every pair of elements of F commute.
   (a) For each $u \in F$, and each eigenvalue λ of u, prove that the eigenspace of u belonging to λ is closed under each endomorphism of F.
   (b) Prove that the endomorphisms of F have a common eigenvector (with possibly different eigenvalues for different $u \in F$).
   (c) Prove that there exists a basis of E consisting of common eigenvectors of F (in other words the endomorphisms of F are *simultaneously diagonalizable*).
5. (5 + 5 pts) Let X be the complex matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix}.$$
   Determine the rational canonical form and the Jordan canonical form of X.
6. (5 + 5 + 5 pts) Let E be a n-dimensional complex vector space and u, v are nilpotent endomorphisms of E.
   (a) What is the characteristic polynomial of u ?
   (b) Suppose that $u \circ v = v \circ u$. Prove that $u + v$ is also nilpotent.
   (c) Prove that $\mathrm{Id}_E + u$ is invertible.

7. (5 + 5 + 5 pts) Let E be a finite dimensional complex inner product space and $u \in \mathrm{End}(E)$.
   (a) Define the adjoint of u and prove that it exists.
   (b) Prove that if u is self-adjoint then the eigenvalues of u are all real.
   (c) Prove that if u is self-adjoint then u is diagonizable.
8. (10 pts) Let E be a finite dimensional complex vector space and $u, v \in \mathrm{End}(E)$. Let $[u, v] = u \circ v - v \circ u$. Prove that if $\mathrm{rank}([u, v]) \leq 1$, then u and v have a common eigenvector.