Math 55400 **Qualifying Exam** **January 4, 2019** **W. Heinzer**

PUID: ______________________________

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	21	
4	21	
5	20	
6	20	
7	20	
8	16	
9	22	
10	16	
11	24	
Total	200	

1. (20 pts) Let V be an abelian group generated by elements a, b, c. Assume the following relations hold: $2a = 4b, 2b = 4c, 2c = 4a$, and these three relations generate all the relations on a, b, c.

(a) Write down a relation matrix for V.

(b) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by x, y, z, and express your generators x, y, z in terms of a, b, c.

(c) What is the order of V?

(d) What is the order of the element a?

2. (21 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space over a field F. Let $c_1, \ldots, c_k$ be distinct elements in F and let $p(x) = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ be the minimal polynomial of T. Let $W_i = \{v \in V \mid (T - c_i I)^{r_i}(v) = 0\}$.

(a) Describe linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) Describe how to obtain linear operators D and N such that $T = D + N$, where D is diagonalizable, N is nilpotent and D and N are polynomials in T.

(c) If $T = D' + N'$, where D' is diagonalizable and N' is nilpotent and $D'N' = N'D'$, prove that $D = D'$ and $N = N'$.

3. (21 pts) Let notation be as in the previous problem and let $f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ be the characteristic polynomial for T. Thus $n = d_1 + \cdots + d_k$ and $1 \leq r_i \leq d_i$ for each i.

(a) If $r_i + 1 = d_i$ for each $i \in \{1, \ldots, k\}$, describe the Jordan form for T.

(b) If $r_i + 2 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible for T?

(c) If $r_i + 3 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible for T?

4. (10 pts) Let V be a finite-dimensional vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

5. (10 pts) Let V be a finite-dimensional vector space over an infinite field F and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in V. Prove that there exists a linear functional f on V such that $f(\alpha_i) \neq 0$ for each i with $1 \leq i \leq m$.

6. (20 pts) Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) (2 pts) Define the adjoint T^* of T.

(b) (6 pts) If $T = T^*$, prove that every characteristic value of T is a real number.

(c) (6 pts) Assume that $T = T^*$ and that c and d are distinct characteristic values of T. If α and β in V are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that α and β are orthogonal.

(d) (6 pts) State true or false and justify: If $A \in \mathbb{R}^{5\times 5}$ is symmetric, then A is diagonalizable.

7. Let M be a module over the integral domain D. A submodule N of M is *pure* in M if the following holds: given $y \in N$ and $a \in D$ such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) (10 pts) Let N be a submodule of M and for $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M, and $\operatorname{ann} \overline{x} = \{a \in D \mid a\overline{x} = 0\}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal (d).

(b) (10 pts) If $M = \langle \alpha \rangle$ is a cyclic $\mathbb{Z}$-module of order 12, list the submodules of M and indicate which of the submodules of M are pure in M.

8. (16 pts) Let M be a finitely generated module over the polynomial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

9. (12 pts) Prove or disprove: if V is a vector space over a field F and $T : V \to V$ is a linear operator such that every subspace of V is invariant under T, then T is a scalar multiple of the identity operator.

10. Let F be a field and let $g(x) \in F[x]$ be a monic polynomial.

(a) (5 pts) Describe the $F[x]$-submodules of $V = F[x]/(g(x))$.

(b) (5 pts) If $g(x) = x^3(x-1)$, diagram the lattice of $F[x]$-submodules of $V = F[x]/(g(x))$.

11. (16 pts) Classify up to similarity all 3×3 complex matrices A such that $A^3 = I$, the identity matrix. How many equivalence classes are there?

12. (8 pts) Let V be an abelian group with generators (v_1, v_2, v_3) that has the matrix $\begin{bmatrix} 2 & 0 & 6 \\ 6 & 12 & 0 \end{bmatrix}$ as a relation matrix. Express V as a direct sum of cyclic groups.

13. (16 pts) Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relation matrix for V as a $\mathbb{Z}$-module.

(b) Let W be the cyclic subgroup of V generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$. Write down a relation matrix for W.

(c) Write down a relation matrix for the quotient module V/W.

(d) What is the cardinality of the quotient module V/W?