Math 55400 **Qualifying Exam** **August 2016** **W. Heinzer**

PUID: ____

Instructions:

1. The point value of each exercise occurs to the left of the problem.
2. Write your answer in the box, if one is provided.
3. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	24	
3	20	
4	16	
5	16	
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7	20	
8	20	
9	20	
10	20	
11	20	
Total	200	

1. (24 pts) Let F be a finite field with p elements, let V be a 3-dimensional vector space over F and let $T : V \to V$ be a linear operator that has minimal polynomial x^2.

(a) (6 pts) How many 1-dimensional T-invariant subspaces does V have? Explain.

(b) (6 pts) How many 1-dimensional T-invariant subspaces W of V are direct summands of V, i.e., are such that $V = W \oplus W'$, where W' is a T-invariant subspace of V?

(c) (6 pts) How many 2-dimensional T-invariant subspaces does V have? Explain.

(d) (6 pts) How many 2-dimensional T-invariant subspaces are direct summands of V?

2. (10 pts) Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

3. (10 pts) Let V be a finite-dimensional vector space over an infinite field F and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in V. Prove that there exists a linear functional f on V such that $f(\alpha_i) \neq 0$ for each i with $1 \leq i \leq m$.

4. (16 pts) Let M be a module over the integral domain D. A submodule N of M is *pure* in M if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) (8 pts) If N is a direct summand of M, prove that N is pure in M.

(b) (8 pts) For $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is pure in M, and $\operatorname{ann} \overline{x} = \{a \in D \mid a\overline{x} = 0\}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal (d).

5. (16 pts) Let F be a field and let M be a finitely generated module over the polynomial ring $F[x]$. Let N be a submodule of M that is pure in M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

6. (12 pts) Let F be a field.

(a) What is the dimension of the vector space of all 3-linear functions $D : F^{3\times 3} \to F$? Explain.

(b) What is the dimension of the vector space of all 3-linear alternating functions $D : F^{3\times 3} \to F$? Explain.

7. (6 pts) If $t_0, t_1, \ldots, t_n$ are $n+1$ distinct elements of a field F and $c_0, c_1, \ldots, c_n$ are elements of F, write down a polynomial $g(x) \in F[x]$ of degree $\leq n$ such that $g(t_i) = c_i$ for each $i \in \{0, 1, \ldots, n\}$.

8. (6 pts) Let $\mathbb{Q}$ denote the field of rational numbers. Give an example of a linear operator $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ having the property that the only T-invariant subspaces are the whole space and the zero subspace. Explain why your example has this property.

9. (20 pts) Let $A \in \mathbb{C}^{5\times 5}$ be a diagonal matrix with exactly four distinct entries on its main diagonal.

(a) (6 pts) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in A?

(b) (6 pts) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{5\times 5}$ such that $AB = BA$?

(c) (8 pts) If $B \in \mathbb{C}^{5\times 5}$ is a diagonal matrix with exactly four distinct entries on its main diagonal, is B similar to a polynomial in A? Justify your answer.

10. (20 pts) Let V be an abelian group generated by elements a, b, c. Assume that $2a = 4b, 2b = 4c, 2c = 4a$, and that these three relations generate all the relations on a, b, c.

(a) (4 pts) Write down a relation matrix for V.

(b) (4 pts) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by x, y, z.

(c) (4 pts) Express your generators x, y, z in terms of a, b, c.

(d) (8 pts) List the orders of elements in V and the number of elements of each order.

11. (20 pts) Let V be a finite-dimensional vector space over an infinite field F and let $T : V \to V$ be a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that V is a direct sum of cyclic $F[x]$-modules.

(b) In terms of the expression for V as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that V have only finitely many T-invariant subspaces? Explain.

12. (20 pts) Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) Define the adjoint T^* of T.

(b) If $T = T^*$, prove that every characteristic value of T is a real number.

(c) Assume that $T = T^*$ and that c and d are distinct characteristic values of T. If α and β in V are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that α and β are orthogonal.

(d) If W is a T-invariant subspace of V, prove or disprove that the orthogonal complement $W^\perp$ must be T^*-invariant.

13. (20 pts) Let $A \in \mathbb{C}^{3\times 3}$ be a diagonal matrix with main diagonal entries $1, 2, 3$. Define $T_A : \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}$ by $T_A(B) = AB - BA$.

(a) What is the dimension of the null space of T_A?

(b) What is the dimension of the range of T_A?

(c) What are the characteristic values of T_A?

(d) What is the minimal polynomial of T_A?

(e) Is T_A diagonalizable? Explain.