Math 55400 **Qualifying Exam** **August, 2015** **W. Heinzer**

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Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	20	
5	20	
6	20	
7	15	
8	13	
9	18	
10	20	
11	18	
12	16	
Total	200	

1. (20 pts) Let p be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be a vector space over F and $T : V \to V$ a linear operator. Assume that T has characteristic polynomial x^3 and minimal polynomial x^2.

(a) Express V as a direct sum of cyclic $F[x]$-modules.

(b) How many non-cyclic 2-dimensional T-invariant subspaces does V have?

(c) How many 2-dimensional T-invariant subspaces of V are direct summands of V?

(d) How many 1-dimensional T-invariant subspaces does V have?

(e) How many 1-dimensional T-invariant subspaces of V are not direct summands of V?

2. Let V be a finite-dimensional vector space over a field F, let $T : V \to V$ be a linear operator, and let $p(x) \in F[x]$ be the minimal polynomial of T. Assume that $p(x) = p_1^{r_1} \cdots p_k^{r_k}$, where the $p_i \in F[x]$ are distinct monic irreducible polynomials, $i = 1, \cdots, k$, and the r_i are positive integers. Let $W_i = \{\alpha \in V \mid p_i(T)^{r_i}(\alpha) = 0\}$.

(a) (10 pts) Describe how to obtain linear operators $E_i : V \to V$, $i = 1, \dots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_iE_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) (10 pts) If $p(x)$ is a product of linear polynomials, describe how to obtain a diagonalizable operator D and a nilpotent operator N such that $T = D + N$, where D and N are both polynomials in T.

3. (20 pts) Let V be a finite-dimensional vector space over an infinite field F and let $T : V \to V$ be a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that V is a direct sum of cyclic $F[x]$-modules.

(b) In terms of an expression for V as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that V have only finitely many T-invariant subspaces? Explain.

4. (20 pts) Let V be a finite-dimensional vector space over a field F and let W_1, W_2 and W_3 be nonzero subspaces of V.

(a) If $W_1 \cap W_2 = 0$, prove or disprove that every vector β in $W_1 + W_2$ has a unique representation as $\beta = \alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

(b) If $W_i \cap W_j = 0$ for each $i \neq j$ with $i, j \in \{1, 2, 3\}$, prove or disprove that every vector β in $W_1 + W_2 + W_3$ has a unique representation as $\beta = \alpha_1 + \alpha_2 + \alpha_3$, where $\alpha_i \in W_i$, $1 \leq i \leq 3$.

5. (20 pts) Let D be a principal ideal domain, let n be a positive integer, and let $D^{(n)}$ denote a free D-module of rank n.

(a) If L is a submodule of $D^{(n)}$, prove that L is a free D-module of rank $m \leq n$.

(b) If L is a proper submodule of $D^{(n)}$, prove or disprove that rank $L < n$.

6. (15 pts) Let M be a module over an integral domain D. A submodule N of M is *pure* in M if for every $y \in N$ and $a \in D$ the following condition holds: if $ax = y$ for some $x \in M$, then there exists $z \in N$ with $az = y$.

(a) If $M = \langle m \rangle$ is a cyclic $\mathbb{Z}$-module of order 24, list all the pure submodules of M.

(b) For a submodule N of M and $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in M/N. Prove that $\operatorname{ann} \overline{x} := \{a \in D \mid a\overline{x} = 0\} \supseteq \operatorname{ann} x := \{a \in D \mid ax = 0\}$.

(c) If N is pure in M, and $\operatorname{ann} \overline{x}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = (d)$.

7. (13 pts) Let M be a finitely generated module over the polynomial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

8. (18 pts) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V and let $R = T(V)$ denote the range of T.

(a) Prove that R has a complementary T-invariant subspace if and only if R is independent of the null space N of T, i.e., $R \cap N = 0$.

(b) If R and N are independent, prove that N is the unique T-invariant subspace of V that is complementary to R.

9. (20 pts) Let A and B be in $\mathbb{Q}^{n\times n}$ and let $I \in \mathbb{Q}^{n\times n}$ denote the identity matrix.

(a) State true or false and justify: if A and B are similar over an extension field F of $\mathbb{Q}$, then A and B are similar over $\mathbb{Q}$.

(b) Let M and N be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[x]$. Define "M and N are equivalent over $\mathbb{Q}[x]$."

(c) State true or false and justify: If $\det(xI - A) = \det(xI - B)$, then $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$.

(d) State true or false and justify: If $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$, then A and B are similar over $\mathbb{Q}$.

10. (18 pts) Let $A \in \mathbb{C}^{4\times 4}$ be a diagonal matrix with exactly three distinct entries on its main diagonal.

(a) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in A?

(b) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{4\times 4}$ such that $AB = BA$?

(c) If $B \in \mathbb{C}^{4\times 4}$ is a diagonal matrix with exactly three distinct entries on its main diagonal, is B similar to a polynomial in A? Justify your answer.

11. (8 pts) Let V be an abelian group with generators (v_1, v_2, v_3) that has the matrix $\begin{bmatrix} 4 & 0 & 8 \\ 4 & 12 & 0 \end{bmatrix}$ as a relation matrix. Express V as a direct sum of cyclic groups.

12. (8 pts) Let V be an abelian group with generators (v_1, v_2, v_3) that has the matrix $\begin{bmatrix} 4 & 0 & 8 \\ 4 & 12 & 0 \\ 2 & 2 & 0 \end{bmatrix}$ as a relation matrix. Express V as a direct sum of cyclic groups.