LINEAR ALGEBRA COMPREHENSIVE EXAM – JAN, 2012

Attempt all questions. Time 2 hrs

1. Let A be a commutative ring with identity, and let E be a free A-module of finite rank, and $u \in \mathrm{End}_A(E)$.
   (a) (2 pts) Give a basis free definition of $\det(u)$.
   (b) (8 pts) Prove that the following are equivalent.
      (i) u is bijective.
      (ii) u is surjective.
      (iii) $\det(u)$ is invertible in A.

2. Let k be an algebraically closed field, V a finite dimensional k-vector space and $u \in \mathrm{End}(V)$.
   (a) (2 pts) Define the minimal polynomial of u.
   (b) (6 pts) Prove that u is diagonalizable if and only if its minimal polynomial has simple roots.
   (c) (2 pts) Is the same true if we replace "minimal polynomial" by the characteristic polynomial ? Justify your answer.

3. Let k be a field, V an n dimensional vector space and $u \in \operatorname{End}(V)$.
   (a) (2pts) Define the similarity invariants, $q_1, \ldots, q_n$ of u.
   (b) (4 pts) Prove that the characteristic polynomial χ_u is equal to the product $q_1 \cdots q_n$.
   (c) (2pts) What are the similarity invariants for the identity and the zero endomorphisms?
   (d) (2pts) Classify upto similarity all $n \times n$ complex matrices A, such that $A^n = 0$.

4. Let V be an n-dimensional complex inner product space.
   (a) (2 pts) Prove that there exists an orthonormal basis of V.
   (b) (2 pts) Define unitary transformations in a basis free way.
   (c) (2 pts) Define the adjoint, v^* of an endomorphism $v \in \mathrm{End}(V)$ is a basis free way.
   (d) (2 pts) Show that if U is an unitary transformation of V, then $uu^* = \mathrm{Id}_n$.
   (e) (2 pts) Prove that if $v \in \mathrm{End}(V)$ and W is a subspace of V closed under v, then $W^\perp$ is closed under v^*.

5. Let k be an algebraically closed field and let V be a finite dimensional k vector space.
   (a) (2 pts) State completely (but do not prove) the (additive) Jordan decomposition theorem.
   (b) (4 pts) Let $u = u_s + u_n, v = v_s + v_n$ be the Jordan decomposition of two commuting endomorphisms $u, v \in \mathrm{End}(V)$. Deduce the Jordan decompositions of $u+v$ and uv. Justify your answer using the statement in the previous part.
   (c) (2 pts) State the multiplicative Jordan decomposition theorem.
   (d) (2 pts) If $v \in \mathrm{End}(V)$ is unipotent, then what is the characteristic polynomial of v ? Justify your answer.